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On the Consistency of the Failure of Square

DISSERTATION

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Abstract of the Dissertation

Square principles are statements about an important class of infinitary combinatorial objects. They may hold or fail to hold at singular cardinals depending on our large cardinal assumptions, but their precise consistency strengths are not yet known.

In this paper I present two theorems which greatly lower the known upper bounds of the consistency strengths of the failure of several square principles at singular cardinals. I do this using forcing constructions. First, using a quasicompact* cardinal I construct a model of $\neg\Box(\aleph_{\omega+1}, < \omega)$. Second, using a cardinal which is both subcompact and measurable, I construct a model of $\Box_{\kappa,2} + \neg\Box_\kappa$ in which κ is singular. This paves the way for several natural extensions of these results.

Introduction

0.1 History

Modern set theory is the study of axioms of mathematics cast in the language of sets. In 1900, David Hilbert published twenty-three outstanding problems in mathematics. Of particular importance to us, his first problem asked if George Cantor's continuum hypothesis (CH) that $2^{\aleph_0} = \aleph_1$ was true, and his second problem asked if the axioms of arithmetic could be shown to be consistent.

In 1931, Kurt Gödel proved his groundbreaking incompleteness theorems which provided an answer to Hilbert's second problem and brought to light the possibility that CH was independent from ZFC. The theorems showed that sufficiently strong systems of axioms for arithmetic can not prove their own consistency, nor can they be both complete and consistent. In 1940, Gödel [6] described his constructible universe L, and showed that L was a model of ZFC + CH. That is, he showed it is consistent with ZFC that the continuum hypothesis is true.

In 1964, Paul Cohen [1, 2] completed Hilbert's first problem by proving that it was consistent ZFC that CH could fail. To do so, Cohen invented a method of building outer models called forcing. This paved the way for many further independence results.

A critical object used in studying axioms is the large cardinal. Large cardinals are infinite cardinal numbers with additional properties that are so strong that their existence and nonexistence cannot be proved from ZFC. They are inaccessible, and so the mathematical universe below any such cardinal will be a model of ZFC. A prime example is Ulam's [9] measurable cardinal, a type of cardinal number which arose from measure theory. Like other large cardinals, measurable cardinals have both a combinatorial definition and a model-theoretic definition. This second approach, introduced by Dana Scott [9], involves maps known as elementary embeddings which connect models in a way that preserves formulas. Thus if a large cardinal exists, we may sometimes use a derived elementary embedding to prove consistency results.

Large cardinals form a natural hierarchy of consistency strength with which we can compare various set-theoretic principles. For example, the large cardinal notion of weak compactness for a cardinal κ is equiconsistent with κ being an inaccessible cardinal at which all κ-trees must have cofinal branches.

0.2 Singular cardinal combinatorics

My research works with infinite combinatorics at singular cardinals. Specifically I have investigated the consistency strength of the failure of several of the so-called "square principles" at singular cardinals and successors to singular cardinals. The first such principle was isolated by Ronald Jensen [8] in his construction of Suslin trees in L. In fact, he showed that $L \models (\forall\kappa)\square_\kappa$, that is, every model of ZFC will have an inner model in which $\square_\kappa$ holds. However, the existence (and equivalently, nonexistence) of square sequences is known to be independent of ZFC. For example, if a model of ZFC contains a supercompact cardinal κ then $\square_\lambda$ will fail to hold in that model.

Although many results exist showing that the existence of supercompact cardinals will cause the failure of various square principles, supercompactness is far higher in consistency strength than one needs. In particular, to get $\square_\kappa$ to fail one only needs to influence the universe up through κ^+, whereas a supercompact cardinal influences all of the universe.

Jensen isolated a property implying the failure of $\square_\kappa$ called subcompactness, which is much weaker than 2^κ-supercompactness. There is evidence suggesting that this is a very promising large cardinal axiom. For example, it was proved by Schimmerling and Zeman [11] that in all core models $\square_\kappa$ fails if and only if κ is subcompact.

However, in all of these examples if we begin with a large cardinal we are by definition working with a regular cardinal. Proving analogous results at singular cardinals is difficult. The general strategy will be to first look for a large cardinal notion at which the desired result holds. For example, we may begin with a supercompact cardinal κ since we know that in a model of such a cardinal $\square_\kappa$ already fails. We then will attempt to build a model in which κ has been made singular, doing so in such a way that $\square_\kappa$ still fails in the target model.

0.3 Research

0.3.1 Square at a successor to a singular cardinal

In my first theorem I showed that the consistency strength of the failure of $\square(\aleph_{\omega+1}, < \omega)$ can be lowered to that of a slight strengthening of quasicompactness. Kypriotakis [10] proved that it is consistent that $\square(\kappa^+)$ holds at a subcompact cardinal κ (modulo the existence of an extender model), so subcompactness is unlikely to be enough to prove such a result. However, if κ is a quasicompact cardinal then $\square(\kappa^+)$ fails, so this is where we begin our proof, with a slight strengthening of quasicompactness which has appeared to be necessary for our forcing construction.

Theorem 0.3.1 (Holben) *(GCH) It is consistent relative to the existence of a quasicompact* cardinal that* $\square(\aleph_{\omega+1}, < \omega)$ *fails.*

Proofs of this kind often follow a standard pattern. One often will use an elementary embedding to push a coherent κ^+-sequence of squares up, creating a longer sequence. We can then pull back a club set from the new sequence and show that it coheres with our original sequence, thus proving that the square principle cannot hold in the original model containing the large cardinal. This sort of approach works for many types of square principles and a number of large cardinal axioms.

As mentioned above, since large cardinals are regular we must work harder to do the same kind of proof while turning κ into a singular cardinal. To do so one may have to do a forcing construction, such as Prikry forcing. Prikry forcing [5] is the standard tool for turning a measurable cardinal into a singular. This complicates our proof, however, as we must work with names for sets rather than the sets themselves. Additionally, we must construct all of our arguments in a way such that our desired large cardinal properties, such as $\neg\Box(\kappa^+, < \omega)$, will still hold after forcing. In this proof we use a modification of Prikry forcing in which we interleave collapse forcings. This will turn κ into $\aleph_\omega$ without collapsing κ^+.

Finally, it should be observed that our result is the best possible using our methods, in that we would not expect to get $\neg\Box(\aleph_{\omega+1}, \omega)$ using a Prikry forcing. This follows from the theorem of Cummings and Schimmerling [4] which says that any outer model of V which sees a V-inaccessible cardinal κ as ω-cofinal but agrees on the cardinal successor of κ must contain a $\Box_{\kappa,\omega}$ sequence.

0.3.2 Separation of square principles

The next theorem works towards greatly lowering the consistency strength of separating out the principles $\Box_{\kappa,n}$ at a singular cardinal. It follows from their definitions that $\Box_{\kappa,n}$ implies $\Box_{\kappa,n+1}$, but it can be shown that there exist models in which these principles are not equivalent. The following theorem which says the opposite implication can fail was proved in 2001 [3]:

Theorem 0.3.2 (Cummings-Foreman-Magidor) *Let κ be a supercompact cardinal and suppose that $2^{\kappa^{+\omega}} = \kappa^{+\omega+1}$. Let $1 \leq \mu < \nu < \aleph_\omega$ be two cardinals. Then there is a generic extension satisfying $\Box_{\aleph_\omega,\nu} + \neg\Box_{\aleph_\omega,\mu}$.*

I proved the result for $\Box_{\kappa,2} + \neg\Box_\kappa$ at a singular cardinal. However, our large cardinal assumptions are again much weaker than a supercompact. Additionally, our proof may generalize in several ways which we discussed later. Our proof uses an Easton support iteration to prepare a model in which we can do a Prikry forcing. The proof also adopts a method of Jensen [7] which was originally used to show separation at a regular cardinal.

Theorem 0.3.3 (Holben) *(GCH) It is consistent relative to the existence of a subcompact cardinal which is also measurable that $\square_{\kappa,2}$ holds while $\square_\kappa$ fails at a singular cardinal κ.*

Chapter 1

Background Material

1.1 Large cardinals

Large cardinals are infinite cardinals numbers whose existence is independent of ZFC. Here we define the large cardinal notions relevant to our arguments. First we give a definition for measurability using elementarity.

Definition 1.1.1 *A cardinal κ is measurable if and only if it is the critical point of an elementary embedding $j : V \to M$.*

The definition of measurability is equivalent to a combinatorial construction which we reference here and define in the next section.

Lemma 1.1.1 *A cardinal κ is measurable if and only if there exists a non-principle κ-complete ultrafilter U on κ. If $j : V \to M$ has critical point κ, then $U = \{x \subseteq \kappa \mid \kappa \in j(x)\}$ is such an ultrafilter, and if U is such an ultrafilter, then $j : V \to \mathrm{Ult}(V, U)$ is an elementary embedding with critical point κ, where $\mathrm{Ult}(V, U)$ is the ultrapower of V by U, which we define in the next section.*

Many results can be proved using supercompact cardinals, so we define them here.

Definition 1.1.2 *A cardinal κ is supercompact if and only if for every cardinal $\lambda \geq \kappa$ there exists an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$.*

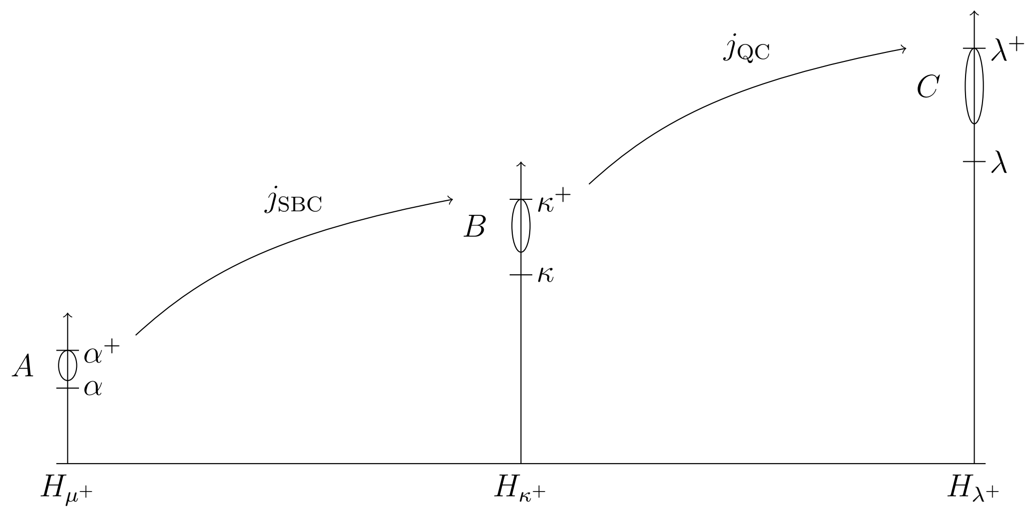

Figure 1.1: Subcompact and quasicompact cardinals.

The results that we will prove rely on subcompact and subcompact cardinals, whose definitions are symmetric .

Definition 1.1.3 *A cardinal κ is subcompact if and only if for every set $B \subseteq H_{\kappa^+}$ there exists $\alpha < \kappa$, $A \subseteq H_{\alpha^+}$ and an elementary embedding*

$$j : \langle H_{\alpha^+}, \in, A\rangle \to \langle H_{\kappa^+}, \in, B\rangle$$

such that $\mathrm{crit}(j) = \alpha$.

Definition 1.1.4 *A cardinal κ is quasicompact if and only if for every set $B \subseteq H_{\kappa^+}$ there exists $\lambda > \kappa$, $C \subseteq H_{\lambda^+}$ and an elementary embedding*

$$j : \langle H_{\kappa^+}, \in, B\rangle \to \langle H_{\lambda^+}, \in, C\rangle$$

such that $\mathrm{crit}(j) = \kappa$.

Both of the results in this paper assume GCH, so the following lemma is relevant:

Lemma 1.1.2 *If GCH holds, then the above formulations of subcompactness and quasicompactness are equivalent to formulations where we take subsets of the form $A \subseteq \gamma^+$ instead of $A \subseteq H_{\gamma^+}$ for cardinals γ.*

1.2 Some models of set theory

We will use several standard types of models in proving consistency results.

Definition 1.2.1 *For any regular cardinal θ let*

$$H_\theta = \{x \mid |\mathrm{trcl}(x)| < \theta\}$$

be the set of all sets with transitive closure of size less than θ.

Lemma 1.2.1 *For any regular uncountable cardinal θ, $H_\theta \models \mathrm{ZFC}^-$.*

Whenever we have a measure U we may construct a useful model of set theory called the ultrapower.

Definition 1.2.2 (Ultrapower) *Let κ be measurable as witnessed by the ultrafilter U on κ. We define an equivalence relation on ${}^\kappa V$ by $f =_U g$ if and only if $\{\xi \in \kappa \mid f(\xi) = g(\xi)\} \in U$. In order to make sure our equivalence classes are sets, we require declare that for any $f \in {}^\kappa V$, $[f]_U = \{g \in {}^\kappa V \mid g =_U f \wedge g$ is of minimum rank$\}$. We then let $\mathrm{Ult}(V, U) = \{[f]_U \mid f \in^\kappa V\}$ be the ultrapower of the universe by U.*

Theorem 1.2.1 (Łoś) *Let $\varphi(v_1, ...v_n)$ be a formula of set theory and let $f_1, ...f_n \in^\kappa V$. Then*

$$\mathrm{Ult}(V, U) \models \varphi([f_1]_U, ..., [f_n]_U)$$

if and only if

$$\{\xi \in \kappa \mid \varphi(f_1(\xi), ..., f_n(\xi))\} \in U.$$

1.3 Square Principles

Square sequences are sequences of club sets which cohere with each other, taken with some additional properties. They were originally isolated by Jensen in his construction of Suslin trees in the model L. There are many variants on square, and we list those relevant to this paper below.

Definition 1.3.1 *Let α be a limit ordinal and let $C \subseteq \alpha$ be a set.*

- *We say that C is club in α if and only if it contains all of its limit points below α and it is unbounded in α.*

- *For $\kappa < \alpha$ we say that C is $< \kappa$-club in α if and only if C contains all of its limit points below α of cofinality $< \kappa$, and is unbounded in α.*

Definition 1.3.2 *A set S is stationary in α if and only if it has nonempty intersection with every set club in α.*

We are now ready to define the most basic kind of square sequence.

Definition 1.3.3 *Let κ be a cardinal, and let $C = \langle C_\alpha \mid \alpha \in \lim(\kappa, \kappa^+) \rangle$ be a sequence such that for each $\alpha \in \lim(\kappa, \kappa^+)$*

- *C_α is club in α,*

- *$\mathrm{otp}(C_\alpha) \leq \kappa$, and*

- *(Coherency) $\beta \in \lim(C_\alpha) \to C_\beta = \beta \cap C_\alpha$.*

C is called a $\square_\kappa$ sequence, and if such a sequence exists we say that $\square_\kappa$ holds.

We now define what it means to be a thread for a coherent sequence of club sets. The definition can be naturally generalized for other types of square sequences. The existence of threads will be the primary method by which we will observe the failure of square principles.

Definition 1.3.4 *Let C be a sequence of coherent clubs that is cofinal in κ. Then we say that D is a thread for C if and only if D is a club set in κ which coheres with the sequence.*

Definition 1.3.5 *Let κ be a regular cardinal, and let $C = \langle C_\alpha \mid \alpha \in \lim(\kappa) \rangle$ be a sequence such that,*

- *For every $\alpha \in \lim(\kappa)$,*

  - *C_α is club in α,*

- *(Coherency)* $\beta \in \lim(C_\alpha) \to C_\beta = \beta \cap C_\alpha$, *and*

- *There is no thread for the sequence.*

C is called a $\square(\kappa)$ *sequence, and if such a sequence exists we say that* $\square(\kappa)$ *holds.*

Observe that $\square_\kappa$ implies $\square(\kappa^+)$. They both assert the existence of κ^+ sequences, so the fact that κ and κ^+ do not match up is purely notational. Finally, we will also be working with a weakened version of square formulated by Schimmerling, in which our sequence elements are sets of clubs, rather than clubs themselves. We define this generalization of $\square_\kappa$ below, but the $\square(\kappa)$ version is generalized identically.

Definition 1.3.6 *Let* $\lambda \leq \kappa$ *be cardinals. We say that* $\langle C_\alpha \mid \alpha \in \lim(\kappa, \kappa^+)\rangle$ *is a* $\square_{\kappa,\lambda}$ *sequence if and only if for each* $\alpha \in \lim(\kappa, \kappa^+)$

- $1 \leq |C_\alpha| \leq \lambda$,

- *For all* $C \in C_\alpha$,

  - C *is club in* α,

  - $\mathrm{ot}(C) \leq \kappa$, *and*

  - *(Coherency)* $\beta \in \lim(C) \to C \cap \beta \in C_\beta$.

If such a sequence exists we say that $\square_{\kappa,\lambda}$ *holds.*

Chapter 2

Forcing

Forcing is a method used to construct outer models of set theory which can be used to find upper bounds to the consistency strength of various principles of interest. We will use forcing methods in conjunction with elementary embeddings to produce models in which our results are achieved. Here we list some basic facts as well as the forcing constructions we will use.

2.1 Basics

We begin with some of the combinatorial properties of partially ordered sets, and then we give their immediate consequences in forcing.

Definition 2.1.1 (Combinatorial properties of posets) *Let $(P, <)$ be a forcing poset. Then*

1. *$(P, <)$ has the κ chain condition, or the κ-c.c., if and only if there are no antichains in P of size κ. As a special case, we traditionally say $(P, <)$ has the countable chain condition (c.c.c.) if and only if it has the ω_1-c.c.*

2. *$(P, <)$ is κ-closed if and only if for every descending chain $p_0 \geq p_1 \geq ... \geq p_\xi \geq ...$ of length γ, where $\gamma < \kappa$, there exists a condition p such that for all $\xi < \gamma$ we have $p \leq p_\xi$.*

3. *$(P, <)$ is κ-directed closed if and only if for every directed set $D \subseteq P$ with size $\leq \kappa$ there exists some $p \in P$ such that $p \leq d$ for all $d \in D$.*

4. $(P,<)$ *is strategically* κ*-closed if and only if for every two player game* $\mathcal{G}(P)$ *of length* $\leq \kappa$ *where player I plays on the even and limit steps, player II has a winning strategy.*

5. $(P,<)$ *is* κ*-distributive if and only if the intersection of* $<\kappa$*-many open dense sets is open and dense.*

Theorem 2.1.1 *Let* $(P,<)$ *be a notion of forcing and let* κ *be a cardinal. Then*

1. *If* P *has the* κ*-c.c. then* P *preserves cofinalities* $\geq \kappa$*. If* κ *is regular then* P *preserves cardinals* $\geq \kappa$*.*

2. *If* P *is* κ*-distributive then* P *preserves cofinalities and cardinals* $\leq \kappa$*.*

3. *If* P *is* κ*-directed closed then it is* κ^+*-closed.*

4. *If* P *is* κ^+*-closed then it is strategically* κ*-closed.*

5. *If* P *is* κ*-closed then it is* κ*-distributive.*

We now define a basic kind of forcing which is used in our first result.

Definition 2.1.2 (Lévy Collapse) *Let* $\kappa \leq \lambda$ *be cardinals such that* κ *is regular and* λ *is inaccessible. Let* $p \in Col(\kappa, <\lambda)$ *if and only if*

- $p : \lambda \times \kappa \to \lambda$ *is a partial function such that* $|p| < \kappa$*,*
- *For every* $(\alpha, \beta) \in \mathrm{dom}(p)$*,* $p(\alpha, \beta) < \alpha$*.*

For $p, q \in Col(\kappa, <\lambda)$ *we let* $p \leq$ *if and only if* $p \supseteq q$*.*

Theorem 2.1.2 *Let* G *be a filter for* $Col(\kappa, <\lambda)$ *that is generic over* V*. Then*

$$V[G] \models \lambda = \kappa^+.$$

2.2 Prikry forcing

Here we define a forcing that turns a measurable cardinal into a singular cardinal, invented by Karel Prikry in his doctoral thesis. Forcing with this poset will add an ω-sequence that is cofinal in our cardinal, but will not collapse any cardinals.

Definition 2.2.1 *Let κ be a measurable cardinal with measure U. Let P be the set of all pairs (p, A) where*

1. *p is a finite increasing sequence of ordinals in κ,*
2. *$A \in U$, and*
3. *$min(A) > max(p)$.*

Let $(p, A), (q, B) \in P$. We say that $(p, A) \leq (q, B)$ if and only if

1. *p end extends q,*
2. *$A \subseteq B$, and*
3. *$p \backslash q \subseteq B$.*

We define a subordering of $\leq$ by saying that (p, A) is a direct extension of (q, B), and writing $(p, A) \leq^* (q, B)$, if and only if

1. *$p = q$, and*
2. *$A \subseteq B$.*

The following is a very useful property of Prikry forcing, which says that arbitrary forcing conditions do not have to be extended far in order to decide a statement.

Lemma 2.2.1 (The Prikry condition) *Let φ be a statement in the forcing language of $(P, <)$ and let $(p, A) \in P$. Then there is a direct extension $(q, B) \leq^* (p, A)$ such that*

$$(q, B) || \varphi,$$

where this is read as "(q, B) decides φ", meaning that the condition forces either φ or $\neg\varphi$.

Finally, we show the result of doing a Prikry forcing.

Theorem 2.2.1 *Let G be a filter for $(P, <)$ generic over V. Then in the generic extension $V[G]$, the following holds:*

1. *All cardinals are preserved, and*
2. $\mathrm{cf}(\kappa) = \omega$.

2.3 Modified Prikry forcing

There are many variants on Prikry forcing, depending on if one wants to change the cardinal's cofinality to something larger than ω, if one has different large cardinal assumptions, etcetera. Here we present a modification of ordinary Prikry forcing which can be used to turn a measurable cardinal into $\aleph_\omega$. This is done by interleaving the ordinals in the Prikry sequence with conditions from collapse forcings. By doing so we collapse all of the cardinals inbetween, making κ into the ω^{th} infinite cardinal.

We first define the forcing in a more general setting by not specifying the kinds of forcings we will interleave. The forcing and its properties are fairly involved, so here we only present the definition and some key properties which are used in this paper. First, we begin by assuming we have the following:

- Let κ be a measurable cardinal with measure U and derived ultrapower embedding $i : V \to M \cong \mathrm{Ult}(V, U)$.
- Let $X \in U$ be a subset of inaccessibles below κ, and let $Q : X \to V_{\kappa+1}$ be a function such that for every $\alpha \in X$
  - $Q(\alpha) \subset V_\kappa$ is a α^+-closed forcing,
  - If $\alpha < \beta$ then $Q(\alpha) \cap V_\beta$ is a complete subposet of $Q(\alpha)$.
- There is a filter $F \in V$ that is $i(Q)(\kappa) = [Q]_U$-generic over M.

Given the above, we may now define our forcing.

Definition 2.3.1 *Let $p \in P$ if and only if*

$$p = \langle \delta_0, p_0, ..., \delta_n, p_n, g \rangle$$

and the following properties hold:

- $n \in \omega$,
- *For* $i \leq n$, $\delta_i \in X$,
- *For* $i < n$, $p_i \in Q(\delta_i) \cap V_{\delta_i+1}$, *and* $p_n \in Q(\delta_n)$,
- g *is a function with domain in* U *contained in* X, *such that*
  - *For all* $\alpha \in \mathrm{dom}(g)$ $h(\alpha) \in Q(\alpha)$, *and*
  - $i(g)(\kappa) = [g]_U \in F$.

Suppose that $p, q \in P$ *are defined by*

$$p = \langle \delta_0, p_0, ..., \delta_n, p_n, g \rangle$$

$$q = \langle \epsilon_0, q_0, ..., \epsilon_n, q_n, ..., \epsilon_m, q_m, h \rangle$$

Then we define the ordering $q \leq p$ *by*

- $n \leq q$,
- *For* $i \leq n$, $\delta_i = \epsilon_i$ *and* $q_i \leq_{Q(\delta_i)} p_i$,
- *For* $i > n$, $\epsilon_i \in \mathrm{dom}(g)$ *and* $q_i \leq_{Q(\epsilon_i)} g(\epsilon_i)$,
- $\mathrm{dom}(h) \cap (\epsilon_m, \kappa) \subseteq \mathrm{dom}(g)$, *and*
- *For every* $\alpha \in \mathrm{dom}(h) \cap (\epsilon_m, \kappa)$, $h(\alpha) \leq_{Q(\alpha)} g(\alpha)$.

Finally as we did with ordinary Prikry forcing, we define direct extension. For conditions p *and* q *written as above, let* $p \leq^* q$ *if and only if* $p \leq q$ *and*

- $n = m$, *and*

- *For every i, $\delta_i = \epsilon_i$.*

Now we will discuss some essential properties of this modified Prikry forcing.

Lemma 2.3.1 (The Prikry condition) *Let $p \in P$ and let φ be a statement in the forcing language of P. Then there is some $q \leq^* p$ that decides φ.*

We can now give the main result for this forcing. If we specify the function Q we will be able to conclude more. This will be done when the forcing is used later in this paper.

Theorem 2.3.1 *Let G be generic for P over V. Then*

- *For each $i \in \omega$ there is a unique cardinal δ_i contained in the $(2i)^{\text{th}}$ position of any $p \in G$ with large enough domain. Thus $\vec{\delta} = \langle \delta_i \mid i \in \omega \rangle$ is the well-defined Prikry sequence cofinal in κ added by G, and so $V[G] \models \mathrm{cf}(\kappa) = \omega$.*

- *For each $i \in \omega$ let*

$$G_i = \{p(2i+1) \mid p \in G \text{ and } 2i+1 \in \mathrm{dom}(p)\}$$

  Then G_i is generic for $Q(\delta_i) \cap V_{\delta_{i+1}}$ and is contained in $V[G]$.

2.4 Iteration

It is often useful to force multiple times in succession. If the second forcing depends on how the first generic is picked, and so on, one must use names in order to define the iteration in the ground model. Here we briefly define the basics of iterating forcing posets in this manner.

Definition 2.4.1 (Iterated forcing) *For any ordinal $\alpha \geq 1$ we let $(P_\alpha, <_\alpha)$ denote an iteration of length α. Then we let $p \in P_\alpha$ if and only if p is an α sequence with the following properties:*

1. *If $\alpha = 1$ then for some forcing notion Q_0,*

   (a) P_1 is the set of all 1-sequences $\langle p(0) \rangle$, where $p(0) \in Q_0$, and

(b) $\langle p(0)\rangle \leq_1 \langle q(0)\rangle$ *if and only if* $p(0) \leq q(0)$.

2. *If* $\alpha = \beta + 1$, *then* $P_\beta = P_\alpha \restriction \beta = \{p \restriction \beta \mid p \in P_\alpha\}$ *is an iteration of length* β, *and there is a name* $\dot{Q}_\beta \in V^{P_\beta}$ *for a forcing notion such that*

   (a) $p \in P_\alpha$ *if and only if* $p \restriction \beta \in P_\beta$ *and* $\Vdash_\beta p(\beta) \in \dot{Q}_\beta$, *and*

   (b) $p \leq_\alpha q$ *if and only if* $p \restriction \beta \leq_\beta q \restriction \beta$ *and* $p \restriction \beta \Vdash_\beta p(\beta) \leq q(\beta)$.

3. *If* α *is a limit ordinal, then for every* $\beta < \alpha$, $P_\beta = P_\alpha \restriction \beta$ *is an iteration of length* β, *and*

   (a) *The* α*-sequence* $\langle 1, 1, ..., 1, ...\rangle$ *is in* P_α,

   (b) *If* $p \in P_\alpha$, $\beta < \alpha$, *and* $q \in P_\beta$ *is such that* $q \leq_\beta p \restriction \beta$, *then the condition* $q^\frown(p \restriction [\beta, \alpha))$ *is in* P_α, *and*

   (c) $p \leq_\alpha q$ *if and only if* $p \restriction \beta \leq_\beta q \restriction \beta$ *for all* $\beta < \alpha$.

This definition is unambiguous about what conditions must occur at successor steps, but it leaves us some freedom to choose how we take limit steps. We define the two most common types of limit step definitions below. They are the most extreme possibilities; all iterations at limit ordinals must contain the direct limit and be contained in the inverse limit.

Definition 2.4.2 *Let* P_α *be an iteration and let* $p \in P_\alpha$. *We say the support of* p *is the set* $\text{suppt}(p) = \{\beta < \alpha \mid p \neq 1_\beta\}$.

Definition 2.4.3 *Let* α *be a limit ordinal, and let* P_α *be an iteration. Then*

- P_α *is a direct limit if and only if* $\forall p \in P_\alpha$ p *has bounded support.*
- P_α *is an inverse limit if and only if* $\forall p \in P_\alpha$ $\forall \beta < \alpha$ $p \restriction \beta \in P_\beta$.

Definition 2.4.4 (Easton support) *We say that an iteration* P_α *has Easton support if and only if for every* $p \in P_\alpha$ *and for every regular cardinal* $\gamma \leq \alpha$, $|\text{suppt}(p) \cap \gamma| < \gamma$. *Equivalently, for every limit ordinal* $\gamma \leq \alpha$, P_γ *is a direct limit if* γ *is regular, and* P_γ *is an inverse limit otherwise.*

Lemma 2.4.1 (The factor lemma) *Let $P_{\alpha+\beta}$ be an iteration of $\langle \dot{Q}_\xi \mid \xi < \alpha+\beta \rangle$, where for every limit ordinal $\xi \leq \alpha+\beta$, P_ξ is either a direct or inverse limit. Let $\dot{P}^{(\alpha)}_\beta$ be a name in V^{P_α} for the forcing iteration of $\langle \dot{Q}_{\alpha+\xi} \mid \xi < \beta \rangle$, such that for every limit ordinal $\xi < \beta$, $\dot{P}^{(\alpha)}_\xi$ is a name for a direct or inverse limit, depending on if $P_{\alpha+\xi}$ is a direct or inverse limit.*

Then, if $P_{\alpha+\xi}$ is an inverse limit for every limit ordinal $\xi \leq \beta$ for which $\mathrm{cf}(\xi) \leq |P_\alpha|$, then $P_{\alpha+\beta} \cong P_\alpha * \dot{P}^{(\alpha)}_\beta$.

2.5 Forcing squares

In this section we define two forcings which will be used in our second result. One will add a $\square_{\alpha,2}$ sequence, and one will add a thread for an existing $\square_{\alpha,2}$ sequence.

Definition 2.5.1 *Let α be a regular cardinal. Define P^1_α to be the set of all sequences p such that*

- $\mathrm{dom}(p) = \lim(\delta+1)$ *for some* $\delta \in \lim(\alpha, \alpha^+)$,
- *For all $\delta \in \mathrm{dom}(p)$ $1 \leq |p(\xi)| \leq 2$, and for all $c \in p(\xi)$,*
  - *c is club in ξ,*
  - $\mathrm{ot}(c) \leq \alpha$, *and*
  - *(Coherency) $\beta \in \lim(c) \to c \cap \beta \in p(\beta)$.*

Ordering is by end extension.

Lemma 2.5.1 *The forcing P^1_α is α-distributive, and in particular does not collapse α^+.*

For the moment let us suppose we are in a model M, and let F^1_α be generic for P^1_α over M. Evidently, $\vec{C} = \bigcup F^1_\alpha$ is a $\square_{\alpha,2}$ sequence in $M[G^1_\alpha]$. We may now force again to thread the sequence $\vec{C}$.

Definition 2.5.2 *Let $\vec{C} = \langle C_\xi \mid \xi \in \lim(\alpha, \alpha^+) \rangle$ be a $\square_{\alpha,2}$ sequence. Define P^2_α to be the set of all sets p such that*

- *p is a closed and bounded subset of α^+ with $\mathrm{ot}(p) < \alpha$, and*

- *(Cohereny)* $\beta \in \lim(p) \to p \cap \beta \in \vec{C}_\beta$.

Ordering is by end extension.

Suppose that F^2_α is a generic for P^2_α over V, where P^2_α is built from the $\square_{\alpha,2}$-sequence $\vec{C}$. Then $V[F^2_\alpha] \models \bigcup F^2_\alpha$ is a thread for $\vec{C}$.

Chapter 3

Forcing $\neg\Box(\aleph_{\omega+1}, <\omega)$

In this chapter we will be proving the following theorem which lowers the known consistency strength of the failure of $\Box(\kappa, <\omega)$ at the successor of the first singular cardinal.

Theorem 3.0.1 *(GCH) It is consistent relative to the existence of a quasicompact* cardinal that $\Box(\aleph_{\omega+1}, <\omega)$ fails.*

3.1 Large cardinal assumptions

Our theorem depends on the existence of a modification of quasicompactness which we call quasicompact*, or QC*. It is consistent relative to the existence of a κ^+-supercompact cardinal that a quasicompact* cardinal exists, and it is defined as follows:

Definition 3.1.1 *A cardinal κ is quasicompact* if and only if there exists a cardinal $\lambda > \kappa$ and a map $j : H_{\kappa^+} \to H_{\lambda^+}$ with critical point κ such that for every $B \subseteq \kappa^+$ there is a set $C \subseteq \lambda^+$ for which*

$$j : \langle H_{\kappa^+}, \in, B\rangle \to \langle H_{\lambda^+}, \in, C\rangle,$$

is elementary.

3.2 Defining the forcing

Let P denote a modified Prikry forcing built using a map j and derived ultrafilter U on κ, defined as follows. First, for any condition with no cardinals, the stem will be of the form $p_{-1} \in \mathrm{Col}(\omega, < \kappa)$. All other conditions will begin with a condition $p_{-1} \in \mathrm{Col}(\omega, < \delta_0)$, where δ_0 is the first cardinal in the stem. For the remainder of the stem, we interleave collapses via the function $Q(\alpha) = \mathrm{Col}(\alpha^{++}, < \kappa)$. Finally, let $X = \{\xi < \kappa \mid \xi \text{ is inaccessible}\}$, and let $F \in V$ be a generic for $j(Q)(\kappa) = \mathrm{Col}(\kappa^{++}, < j(\kappa))$ over $M = \mathrm{Ult}(V, U)$.

Now towards a contradiction, suppose there is a condition $p \in P$ and a name $\dot{C} \in V^P$ such that

$$p \Vdash_P \dot{C} = \langle C_\xi \mid \xi \in \lim(\kappa, \kappa^+) \rangle \text{ is a } \square(\kappa^+, < \omega) \text{ sequence.}$$

Let G be a filter generic for P over V containing p. Let $P' = j(P)$ and let $\dot{C}' = j(\dot{C})$.

Lastly, let $\beta = \sup j[\kappa^+]$. Since $j : H_{\kappa^+} \to H_{\mu^+}$, μ^+ is regular, and $\mathrm{otp}(\beta) = \kappa^+ < \mu^+$, it follows that $\beta < \mu^+$. We will be using $\dot{C}'_\beta$ to build a thread for $\dot{C}^G$. Observe that by elementarity we have that

$$j(p) \Vdash_{P'} \dot{C}' \text{ is a } \square(\mu^+, < \omega) \text{ sequence.}$$

Therefore $j(p) \Vdash_{P'} \exists \dot{D} \in \dot{C}'_\beta$. By maximality principle let us fix a name $\dot{D} \in V^{P'}$ for a club set in β such that $j(p) \Vdash_{P'} \dot{D} \in \dot{C}'_\beta$.

3.3 Main lemma

A common application of the Prikry condition is showing that Prikry forcing does not add entirely new club subsets of small order type. That is, any club set added by the forcing must contain a club set from the ground model. Here we show that our modification of Prikry forcing retains this property, and in addition we do not add too many club sets of this kind.

First we shall define some notation. Let $S \downarrow p = \{s \mid s \text{ is a stem s.t. } \exists g\colon \langle s, g \rangle \in P \downarrow p\}$ be the set of all stems for conditions below p in P. For any $s \in S \downarrow p$, let $A_s = \{g \mid \langle s, g \rangle \in P\}$ be the collection of all possible top parts for s in P. Observe that for any such s we have that $|A_s| \leq 2^\kappa$. We will now show that, given a stem $s \in S \downarrow p$, we can find a ground model club that is forced to be a subset of $\dot{D}$, and that this club will be the same regardless of the choice of g.

Lemma 3.3.1 (Main lemma) *Let $s \in S \downarrow p$. Then there exists $q_s \in P^{'}$ and a set $D_s \in V$ which is club in β such that $q_s \leq_{P'} j(s, g)$ for every $g \in A_s$, and $q_s \Vdash_{P'} \check{D}_s \subseteq \dot{D}$. In particular, the definitions of q_s and D_s are independent of the choice of g.*

Proof: First we show that the choice of $g \in A_s$ does not matter. Let $s = \langle p_{-1}, \delta_0, p_0, ...\delta_{n-1}, p_{n-1}\rangle \in S \downarrow p$. Note that $|A_s| = 2^\kappa = \kappa^+$, so enumerate A_s by $\langle g^\alpha \mid \alpha < \kappa^+\rangle$. We now move to the j side and work in $P^{'}$. For each $\alpha < \kappa^+$ let $p_n^\alpha = j(g^\alpha)(\kappa)$. So for each α, $\langle s, \kappa, p_n^\alpha, j(g^\alpha)\rangle < j(\langle s, g^\alpha\rangle) = \langle s, j(g^\alpha)\rangle$. To see why extending our stems in $P^{'}$ by κ is possible, recall that we used j and its derived ultrafilter U to construct P. Since $X \in U$ we have that $\kappa \in j(X)$, so κ may occur in Prikry sequences added by forcing with $P^{'}$. Furthermore, for each $\alpha < \kappa^+$ this gives us that $\mathrm{dom}(g^\alpha) \in U$, so $\kappa \in j(\mathrm{dom}(g^\alpha)) = \mathrm{dom}(j(g^\alpha))$, so we may add κ to the stem of each condition $\langle s, j(g^\alpha)\rangle$. Now since we picked each $p_n^\alpha = j(g^\alpha)(\kappa) \in F$ and F is generic for $j(Q)(\kappa) = \mathrm{Col}(\kappa^{++}, < j(\kappa))$ which is κ^{++}-closed, we may find a single condition $p_n \leq p_n^\alpha$ in $j(Q)(\kappa)$ for all $\alpha < \kappa^+$. Thus $\langle s, \kappa, p_n, j(g^\alpha)\rangle \leq \langle s, \kappa, p_n^\alpha, j(g^\alpha)\rangle$ for each α. Finally, since $P^{'}$ is a Prikry forcing on $j(\kappa)$ and $|A_s| = 2^\kappa < j(\kappa)$, there is a single upper part $\bar{g}$ such that for all $\alpha < \kappa^+$ we have $q := \langle s, \kappa, p_n, \bar{g}\rangle \leq \langle s, \kappa, p_n, j(g^\alpha)\rangle$. We will now work below this condition $q \in P^{'}$.

In V let $T = \langle \tau_\xi \mid \xi \in \kappa^+\rangle$ enumerate a club in β. For the remainder of this proof we will argue entirely using $P^{'}$, so we will drop all such subscripts. Additionally, within this proof we will use the following notation which will let us look at the lower and upper halves of conditions. For any condition $r \in P^{'} \downarrow q$ we will say "lower part of r" to mean $r^0 := r \restriction (2n+1)$. Similarly, we will say "upper part of r" to mean $r^1 := r - (2n+1)$. Note that this is distinct from referring to the bottom or top parts of a condition, which refer to the stem and the function, respectively. We will let $\leq^0$ and $\leq^1$ be the corresponding orderings derived from $\leq_{P'}$.

We now define a κ^+-sequence of conditions below q inductively. First, let $q_0 \leq^* q$ be such that $q_0 || \tau_0 \in \dot{D}$. For the successor step, suppose that we have defined q_ξ for some $\xi < \kappa^+$.

Then let $\bar{q}_\xi = q_0^0{}^\frown q_\xi^1$ and then let $q_{\xi+1} \leq^* \bar{q}_\xi$ be such that $q_{\xi+1}||\tau_{\xi+1} \in \dot{D}$. For the limit step, suppose that ξ is a limit ordinal and that we have defined $q_{\bar{\xi}}$ for every $\bar{\xi} < \xi$. Use the κ^{++}-closure of the upper parts of our conditions in $j(P)(\kappa)$ to find $\bar{q}_\xi^1 \leq^{*1} \bar{q}_{\bar{\xi}}^1$ for each $\bar{\xi} < \xi$, and then let $\bar{q}_\xi = q_0^0{}^\frown \bar{q}_\xi^1$. We then let $q_\xi \leq^* \bar{q}_\xi$ be a direct extension such that $q_\xi||\tau_\xi \in \dot{D}$.

We have produced a sequence $\langle q_\xi \mid \xi < \kappa^+\rangle$. However we do not know if all conditions in this sequence are comparable, so we will pass to a subsequence in which they are. To do this, first note that there are only κ-many possible lower parts to a condition in $P^{'}$, but our sequence is of length κ^+. Therefore there is a stationary set $S \subseteq \kappa^+$ and a tuple $\bar{q}^0$ such that $q_\xi^0 = \bar{q}^0$ for every $\xi \in S$.

Claim 1 *If $\xi, \xi^{'} \in S$ then $\xi < \xi^{'} \to q_{\xi'} \leq^* q_\xi$.*

Proof of claim: First note that for $\alpha < \alpha^{'}$ it is true that $\bar{q}_{\alpha'} \leq \bar{q}_\alpha$, since the conditions' lower parts are equal and the upper parts descend by construction. Now let us compare q_ξ and $q_{\xi'}$. Observe that $q_{\xi'}^1 = \bar{q}_{\xi'}^1 \leq^1 \bar{q}_\xi^1$ and $q_{\xi'}^0 = \bar{q}^0 = q_\xi^0$, so $q_{\xi'} \leq q_\xi$. The lengths of the conditions are the same, so the ordering is by direct extension.
$\square$ (Claim)

Thus $\langle q_\xi \mid \xi \in S\rangle$ is a $\leq^*$-descending κ^+-sequence. Since the upper part of our forcing is κ^{++}-closed, there is a single upper part, call it $\bar{q}^1$, such that $\bar{q}^0{}^\frown\bar{q}^1 \leq^* \bar{q}^0{}^\frown q_\xi^1$ for every $\xi \in S$. Call this full condition $q_s = \bar{q}^0{}^\frown\bar{q}^1$. This condition lies below the portion of our sequence indexed by S.

Now we will show that for almost all $\xi \in S$, $q_s \Vdash \tau_\xi \in \dot{D}$. Suppose not. Let $S^{'} = \{\xi \in S \mid q_\xi \Vdash \tau_\xi \notin \dot{D}\}$ and let $A = \{\tau_\xi \mid \xi \in S^{'}\}$. Note that both of these sets are defined in the ground model.

Claim 2 *If $S^{'}$ is stationary then so is A.*

Proof of claim: Let E be any club set in β. Since T is club as well, the intersection $E \cap T$ is club in β. Viewing T as the function $T : \kappa^+ \to \beta$, let $E^{'} = T^{-1}[E \cap T] = \{\xi \in \kappa^+ \mid \tau_\xi \in E\}$. Since $E \cap T$ is unbounded in β, $E^{'}$ is unbounded in κ^+. Now let $\langle \alpha_\xi \mid \xi \in \gamma\rangle$ be an increasing sequence in $E^{'}$, where $\gamma < \kappa^+$. For each $\xi \in \gamma$, $\tau_{\alpha_\xi} \in E$. Since E is closed, $\sup_{\xi<\gamma} \tau_{\alpha_\xi} \in E$, and since T is continuous, $\sup_{\xi<\gamma} \tau_{\alpha_\xi} = \tau_{\sup_{\xi<\gamma} \alpha_\xi}$. Thus $\tau_{\sup_{\xi<\gamma} \alpha_\xi} \in E$, and so $\sup_{\xi<\gamma} \alpha_\xi \in E^{'}$. We

conclude that E' is club in κ^+. Since we assumed that S' was stationary, there exists some $\xi \in S' \cap E'$. But then $\tau_\xi \in E$ and $\xi \in S'$, thus $\tau_\xi \in A$. We therefore see that $A \cap E \neq \emptyset$, so A is stationary.
□ (Claim)

Let us quickly see that our forcing preserves the stationarity of S', and thus the stationarity of A. To see this, factor P' at κ. The lower part is a forcing of size κ which therefore has the κ-c.c., and the upper part of our forcing's $\leq^{*1}$ ordering is κ^{++}-closed, so neither forcing will add a club in β that doesn't contain a club already in the ground model. It follows that any club in β in $V[G]$ meets S', so stationarity is preserved.

Observe then that by forcing below $j(p)$, A is stationary in β while $q_s \Vdash \check{A} \cap \dot{D} = \emptyset$, a contradiction since $\dot{D}$ is forced to be club. Therefore our assumption that S' is stationary was incorrect, and so we must conclude that $S'' := \{\xi \in S \mid q_s \Vdash \tau_\xi \in \dot{D}\}$ is stationary in κ^+.

We are now ready to define our ground model club set. Let

$$D_s = \{\tau_\xi \mid q_s \Vdash \tau_\xi \in \dot{D}\} = \{\tau_\xi \mid \xi \in S''\}.$$

This is a definition in V. Note that $q_s \Vdash \check{D}_s \subseteq \dot{D}$. Unboundedness of D_s follows since S'' is unbounded in κ^+. To see that D is closed, let $\langle \xi_\alpha \mid \alpha < \gamma \rangle$ be an increasing bounded sequence in S''. Then $q_s \Vdash \tau_{\xi_\alpha} \in \dot{D}$ for every $\alpha < \gamma$, so since q_s sees that $\dot{D}$ is a name for a club set, $q_s \Vdash \sup_{\alpha<\gamma} \tau_{\xi_\alpha} \in \dot{D}$. Since we picked our sequence of τ_ξ's to be continuous, $\sup_{\alpha<\gamma} \tau_{\xi_\alpha} = \tau_{\sup_{\alpha<\gamma} \xi_\alpha}$, so $\sup_{\alpha<\gamma} \xi_\alpha \in S''$. Therefore $\sup_{\alpha<\gamma} \tau_{\xi_\alpha} \in D_s$, and so D_s is closed.
□ (Main lemma)

3.4 Construction

3.4.1 Building the indexing set

We are almost ready to build a thread, but first we will build a set I with which to index it. Given any $s \in S$, let $D_s \in V$ be a club in β and $q_s \in P'$ be such that $q_s \Vdash \check{D}_s \subseteq \dot{D}$, as provided by our main lemma. Let

$$I = \bigcap_{s \in S} D_s.$$

Now observe that since $|S| = \kappa$, we are intersection κ-many sets club in β so since the club filter on β is κ^+-closed, the intersection is nonempty and club. Now we will argue that for any limit ordinals of $j^{-1}[I]$, we get coherence in our sequence $\dot{C}$.

Claim 3 *Let $j(\alpha), j(\alpha^{'}) \in lim(I)$ with $\alpha < \alpha^{'}$. Then*

$$\mathscr{D}_{\alpha,\alpha'} = \{q \leq p \mid q \Vdash \exists \dot{F} \in \dot{C}(\alpha^{'}) : \dot{F} \cap \alpha \in \dot{C}(\alpha)\}$$

is dense in P below p.

Proof of claim: Let $q \leq p$, where $q = (s, g)$. By the main lemma there is a corresponding ground model club D_s and a $P^{'}$-condition $q_s \leq^* j(q)$ such that $q_s \Vdash_{P'} \check{D}_s \subseteq \dot{D}$. Thus $q_s \Vdash j(\alpha), j(\alpha^{'}) \in \lim(I) \subseteq \lim(\check{D}_s) \subseteq \lim(\dot{D})$, and so since coherency is forced, $q_s \Vdash \dot{D} \cap j(\alpha) \in \dot{C}^{'}_{j(\alpha)}$ & $\dot{D} \cap j(\alpha) \in \dot{C}^{'}_{j(\alpha')}$. Thus

$$\exists q^{'} \leq^* j(q) : q^{'} \Vdash_{P'} \exists \dot{F} \in \dot{C}^{'}_{j(\alpha')} \text{ s.t. } \dot{F} \cap j(\alpha) \in \dot{C}^{'}_{j(\alpha)}.$$

Therefore by elementarity,

$$\exists q^{'} \leq^* q : q^{'} \Vdash_P \exists \dot{F} \in \dot{C}_{\alpha'} \text{ s.t. } \dot{F} \cap \alpha \in \dot{C}_{\alpha}.$$

□ (Claim)

3.4.2 Building the thread

Now we need to show that we will force that we have a club set that reaches all the way to κ^+ and coheres with our sequence. Let G be a generic for P containing p. Towards a contradiction, suppose that

$$V[G] \models \text{“}(\exists \alpha \in j^{-1}[\lim(I)])(\forall F \in \dot{C}^G_\alpha)$$

$$(A_{\alpha,F} := \{\gamma \in (\alpha, \kappa^+) \cap j^{-1}[\lim(I)] \mid \exists E \in \dot{C}^G(\gamma) : E \cap \alpha = F\} \text{ is bounded in } \kappa^+)\text{”}.$$

Working in $V[G]$, let $\gamma \in (\sup \bigcup \{A_{\alpha,F} \mid F \in \dot{C}^G_\alpha\}, \kappa^+) \cap j^{-1}[\lim(I)]$. Note that since κ^+ is regular and $|\dot{C}^G_\alpha| < \omega$, $\sup \bigcup \{A_{\alpha,F} \mid F \in \dot{C}^G_\alpha\} < \kappa^+$. By the density of $\mathscr{D}_{\alpha,\gamma}$, there exists $K \in \dot{C}^G_\gamma$ such that $K \cap \alpha \in \dot{C}^G_\alpha$. But then $\gamma \in A_{\alpha,K\cap\alpha}$, a contradiction.

Now pick any $\bar{\alpha} \in j^{-1}[\lim(I)]$, and pick $\bar{F} \in \dot{C}^G_{\bar{\alpha}}$ such that $A_{\bar{\alpha},\bar{F}}$ is unbounded in κ^+. We would like to close upwards. However we must take some care, as not all clubs can be closed upwards to κ^+, and also it is possible that there may exist some $\alpha > \bar{\alpha}$ such that $\dot{C}^G_\alpha$ contains multiple clubs that are equal when intersected with $\bar{\alpha}$.

We shall work inductively starting from $\bar{\alpha}$ to build a partial choice function $T : \kappa^+ \to V[G]$. Let $T(\bar{\alpha}) = \bar{F}$. Now given any $\xi \geq \bar{\alpha}$ at which T has already been defined, let ξ' be any ordinal greater than ξ such that $\xi' \in j^{-1}[\mathrm{I}]$ and $\exists F \in \dot{C}^G_{\xi'}$ such that $F \cap \xi = T(\xi)$. By the above such a ξ' always exists, so the induction will continue up to κ^+. Also by the above, there must exist at least one such $F \in \dot{C}^G_{\xi'}$ such that $A_{\xi',F}$ is unbounded in κ^+. Let $T(\xi')$ equal one such F. Finally, observe that this process can be repeated at limit steps by first taking the supremum of the $\mathrm{dom}(T)$ as defined so far, and then finding ξ' above. This completes the induction. Let

$$E = \bigcup \{T(\alpha) \mid \alpha \in \mathrm{dom}(T)\}.$$

Since our induction is of length κ^+, E is unbounded in κ^+. Observe that we have coherence along the clubs enumerated by T, that is for $\alpha, \beta \in \mathrm{dom}(T)$, $\alpha < \beta \to T(\beta) \cap \alpha = T(\alpha)$. To see that this gives us closure, let $\langle \alpha_\xi \mid \xi < \gamma \rangle \subset E$ be an increasing sequence for some $\gamma < \kappa^+$, and let $\alpha = \bigcup_{\xi<\gamma} \alpha_\xi$. By the unboundedness of $\mathrm{dom}(T)$ let $\beta \in \mathrm{dom}(T) - \alpha$. Then by the coherence along T it follows that $\alpha \in T(\beta) \subset E$, and thus E is club in κ^+.

To see that E coheres with $\dot{C}^G$, pick any $\alpha \in \lim(E)$. Observe by the coherence of T that $\alpha \in \mathrm{dom}(T) \to E \cap \alpha = T(\alpha)$. Since $\mathrm{dom}(T)$ is unbounded there exists some $\beta \in \mathrm{dom}(T)$ such that $\alpha \in \lim(T(\beta))$. Therefore by the coherency of our original sequence, $T(\beta) \cap \alpha \in \dot{C}^G_\beta$. Since $E \cap \beta = T(\beta)$ we therefore have that $E \cap \alpha \in \dot{C}^G_\beta$. Thus E is a thread for our $\square(\kappa^+, <\omega)$ sequence, which is a contradiction to the existence of such a sequence.

3.5 Conclusion

We have shown that $V[G] \models E$ is a thread for the sequence $\dot{C}^G$, but this contradicts that we forced $\dot{C}^G$ to be a $\square(\kappa^+, < \omega)$ sequence. Therefore, $V[G] \models \neg\square(\kappa^+, < \omega)$.

Additionally, if $\langle \delta_n \mid n \in \omega \rangle$ is the Prikry sequence added by G, then our forcing collapses δ_0 to $\aleph_1$, and it collapses δ_n to $\aleph_{2n+1}$ for $n \in [1, \omega)$. Therefore in $V[G]$ we have $\kappa = \aleph_\omega$. Also our forcing has the κ^+-c.c, so κ^+ is preserved as the cardinal successor of κ, and therefore becomes $\aleph_{\omega+1}$ in $V[G]$. Thus we conclude $V[G] \models \neg\square(\aleph_{\omega+1}, < \omega)$.

Chapter 4

Separation of $\square_{\kappa,2}$ and $\square_\kappa$ at Singular Cardinals

It follows by the definition that if $\square_{\kappa,m}$ holds for some for $m \geq 1$, then $\square_{\kappa,n}$ holds in the same model for $n \geq m$. However, one may construct models in which the opposite implications do not hold. In other words, the principles $\square_{\kappa,n}$ need not be equivalent for different n in certain models. In this chapter we show this separation of square principles for $\square_{\kappa,2}$ from $\square_{\kappa,1}$ at a singular cardinal using relatively modest large cardinal assumptions.

Theorem 4.0.1 *(GCH) It is consistent relative to the existence of a subcompact cardinal which is also measurable that $\square_{\kappa,2}$ holds while $\square_\kappa$ fails at a singular cardinal κ.*

4.1 Large cardinal assumptions

Let κ be the least subcompact cardinal which is also measurable. By a standard Easton support iteration one can force GCH while preserving these properties, so we may assume that GCH holds as well. Let U be a measure on κ of Mitchell order 0, and let $e : V \to N = \mathrm{Ult}(V, U)$ be the corresponding ultrapower map. Define S_κ to be the set of all ordinals $\xi < \kappa$ such that there is an elementary map j witnessing the subcompactness of κ, for which $\mathrm{crit}(j) = \xi$ and $j(\xi) = \kappa$.

It follows from the subcompactness of κ that S_κ is stationary. Additionally, observe that $\kappa \notin e(S_\kappa)$. If it was, then N would contain a measure on κ, violating our assumption that

$o(U) = 0$. Thus $S_\kappa \notin U$, a fact that will use when showing that our preparation forcing preserves measurability.

4.2 Defining the preparation forcing

Our forcing will occur in two major steps. First we will do a preparation forcing, and get a model of $\square_{\kappa,2} + \neg\square_\kappa$ in which the measurability of κ has been preserved. We will then follow with a Prikry forcing, and show that our desired results hold in the final generic extension.

We will now define an iterated forcing with Easton support. At each $\alpha \in S_\kappa$ we will add a $\square_{\alpha,2}$ sequence and then immediately thread it. At stage κ at the top of our iteration, we will add a $\square_{\kappa,2}$ sequence but do no threading. Let us now define our iteration by induction. For $\alpha \in S_\kappa$ let P^1_α be the forcing which adds a $\square_{\alpha,2}$ sequence, and let $\dot{P}^2_\alpha$ be a name for the forcing which threads the sequence P^1_α will add. $\dot{Q}_\alpha$ be a name for $P^1_\alpha * \dot{P}^2_\alpha$. On the other hand if $\alpha \notin S_\kappa$ let $\dot{Q}_\alpha = \{1\}$.

Now let P_α be the set of all α-sequences $p = \langle p_\beta \mid \beta < \alpha \rangle$ such that

1. For each $\gamma < \alpha$, $p \restriction \gamma \in P_\gamma$ and $\Vdash_\gamma p_\gamma \in \dot{Q}_\gamma$, and

2. If α is a regular cardinal then p has bounded support; that is, $\exists \xi_0$ s.t. $\forall \xi \geq \xi_0$ $p_\xi = 1$.

For $p, q \in P_\alpha$ we define the ordering $\leq_\alpha$ by letting $p \leq_\alpha q$ if and only if

$$(\forall \gamma < \alpha)(p \restriction \alpha \leq_\gamma q \restriction \gamma \text{ and } p \restriction \gamma \Vdash_\gamma p_\gamma \leq_\gamma q_\gamma).$$

Finally, let $P = P_\kappa * \dot{P}^1_\kappa$ define our complete iteration.

Observe by part 2 of our definition that at regular cardinals we take direct limits, and by part 1 at singular ordinals we must have inverse limits, and so our iteration has Easton support. Next, observe that due to absorption each iterate $Q_\alpha = P^1_\alpha * \dot{P}^2_\alpha$ is equivalent to forcing with $\mathrm{Col}(\alpha^+, \alpha)$. It is easy to see that the properties necessary for absorption are satisfied.

In particular, let us observe that by adding a thread C to a $\square_\alpha$ sequence, we must have added a surjection from α to α^+, thereby collapsing the cardinal α^+. This is a critical property of our forcing which we will use often. To see why this is, suppose that $\vec{C} = \langle C_\xi \mid$

$\xi \in (\alpha, \alpha^+)\rangle$ is a $\Box_\alpha$ sequence, let D be a thread for $\vec{C}$, and suppose that $C_\eta = \eta \cap C_\gamma$. Let $f_\gamma : C_\gamma \to \mathrm{ot}(C_\gamma)$ be an order isomorphism. Then $f_\gamma \restriction \eta$ must be an order isomorphism from C_η to $\mathrm{ot}(C_\eta)$. We see therefore that if club sets cohere, so do their order isomorphisms, and so $\bigcup_{\xi \in \lim(D)} f_\xi : D \to \mathrm{ot}(D)$ must be an order isomorphism. Since $\mathrm{ot}(C_\xi) \leq \alpha$ for all $\xi \in (\alpha, \alpha^+)$, we conclude that $\mathrm{ot}(D) = \sup_{\xi \in \lim(D)} \mathrm{ot}(C_\xi) \leq \alpha$. Thus the order type of D is at most α, and so we have a surjection from α onto α^+.

4.3 Lifting the embedding

Let θ be a regular cardinal that is large enough such that $P \in H_\theta$. It follows from subcompactness that there is a set H, an elementary embedding $j : H \to H_\theta$, and a cardinal $\alpha \in S^*_\kappa$ with the properties that $H_{\alpha^+} \subseteq H$, $\mathrm{crit}(j) = \alpha$, $j(\alpha) = \kappa$, and $j(P') = P$ for a set $P' \in H$. Observe that by elementarity we have that $P' = P_\alpha * \dot{P}^1_\alpha$, so we will refer to P' by its factors from now on. Finally, let G_α be a generic for P_α over V.

Let us note here that our general convention in this chapter for sets X in the range of j or any of its liftings is that $j(X') = X$ for some X'. This is because most sets of concern will be first defined on the j-side, and then we will find a corresponding set that our embedding maps to it.

4.3.1 First lifting

We shall first lift j to an elementary map with domain $H[G_\alpha]$. First, observe that since P has Easton support that we can factor P at α. Let us write $P \cong P_\alpha * \dot{Q} * \dot{P}_\kappa$, where $\dot{Q}$ is a name for an Easton support forcing Q by the same definition as P, but with domain $[\alpha, \kappa)$. Let

$$\Delta = \{q \in Q \mid \exists p \in G_\alpha \text{ s.t. } q = \dot{q}^{G_\alpha}, \text{ where } j(p) = (s, \dot{q})\}.$$

This is a definition with parameter G_α, so $\Delta \in H_\theta[G_\alpha]$. Let us show that Δ is a directed set. Let $q_1, q_2 \in Q$, and let $p_1, p_2 \in G_\alpha$ witness this. Since G_α is a filter, there exists $p \in G_\alpha$ such that $p \leq p_1, p_2$. By elementarity, $j(p) \leq j(p_1), j(p_2)$. Writing $j(p) = (s, \dot{q}), j(p_1) = (s_1, \dot{q}_1)$, and $j(p_2) = (s_2, \dot{q}_2)$ this means that $s \Vdash_{P_\alpha} \dot{q} \leq \dot{q}_1, \dot{q}_2$. Finally, observe that in fact $j(p) = (p, \dot{q})$ for some $\dot{q}$. So since $p \in G_\alpha$, $s \in G_\alpha$, and so $\dot{q}^{G_\alpha} \leq q_1, q_2$, meaning that $q \in \Delta$.

Now observe that since the iterates of $\dot{Q}^{G_\alpha}$ are equivalent to $\mathrm{Col}(\xi, \xi^+)$ for $\xi \in [\alpha, \kappa) \cap S_\kappa$, and these forcings are $< \xi$-directed closed, the forcing $\dot{Q}^{G_\alpha}$ is $< \lambda$-directed closed where $\lambda = \min(S_\kappa - \alpha)$. Since $|\Delta| \leq |G_\alpha| \leq |P_\alpha| = 2^\alpha < \lambda$, there exists a single condition $a \in Q$ such that $\forall d \in D\ a \leq d$.

Let K be a generic for $\dot{Q}^{G_\alpha}$ over $H_\theta[G_\alpha]$ such that $a \in K$. Let $G_\kappa = \{(s, \dot{q} \mid s \in G_\alpha \text{ and } \dot{q}^{G_\alpha} \in K\}$ be the generic that factors as $G_\kappa = G_\alpha * K$. Now define $j_1 : H[G_\alpha] \to H_\theta[G_\kappa]$ by

$$j_1(\sigma^{G_\alpha}) = j(\sigma)^{G_\kappa},$$

where $\sigma \in H^{P_\alpha}$. By a standard argument this is a well-defined elementary embedding extending j.

4.3.2 Second lifting

First, let us factor K as $F^1_\alpha * F^2_\alpha * K'$, where F^1_α and F^2_α are generics for P^1_α and P^2_α taken over the appropriate models. We will now lift j_1 to an elementary map with domain $H[G_\alpha * F^1_\alpha]$.

Let $\vec{C}^\alpha = \bigcup F^1_\alpha$ be the $\square_{\alpha,2}$ sequence added by P^1_α and let $D^\alpha = \bigcup F^2_\alpha$ be the thread for $\vec{C}^\alpha$ added by P^2_α. For each limit $\xi \in \alpha^+$ we have that $\vec{C}^\alpha \restriction (\xi + 1) \in H[G_\alpha]$ since these restrictions are conditions in $(\dot{P}^1_\alpha)^{G_\alpha}$. In addition, for each $\xi \in \lim(D^\alpha)$ we have that $D^\alpha \cap \xi \in \vec{C}^\alpha(\xi) \subset H[G_\alpha]$.

Let $\beta = \sup j[\alpha^+]$. Since j is a subcompactness map, $\beta < \kappa^+$. Let $\bar{p} = \bigcup j_1[F^1_\alpha]$ and let $\bar{c} = \bigcup j_1[D^\alpha]$. Then let $a' = \bar{p} \cup \{\langle \beta, \{\bar{c}\}\rangle\}$. This is a condition in $(\dot{P}^1_\kappa)^K$. Let F^1_κ be a generic for $(\dot{P}^1_\kappa)^K$ over $H_\theta[G_\alpha * K]$ such that $a' \in F^1_\kappa$. Now define $j' : H[G_\alpha * F^1_\alpha] \to H_\theta[G_\alpha * K * F^1_\kappa]$ in the usual way, letting

$$j'(\sigma^{F^1_\alpha}) = j_1(\sigma)^{F^1_\kappa},$$

where $\sigma \in H[G_\alpha]^{(\dot{P}^1_\alpha)^{G_\alpha}}$ is any name. Again by a standard argument, $j' \supset j_1 \supset j$ is an elementary embedding. For simplicity let $G = G_\alpha * K * F^1_\kappa$ and let $G' = G_\alpha * F^1_\alpha$, and since we will not need to refer to our original j or the first lifting j_1, from now on we will just write j when we mean j'. Thus we have the elementary embedding $j : H[G'] \to H_\theta[G]$ which we can now use in our arguments.

4.4 Doing the preparation forcing

We shall now look at the generic extension $H_\theta[G]$ and check that $\square_{\kappa,2}$ holds, $\square_\kappa$ does not, and κ remains a measurable cardinal.

First, it is easy to see that $H_\theta[G] \models \square_{\kappa,2}$. The forcing P^1_κ adds such a sequence. By an argument similar to the one presented at the end of section 4.2, if our forcing had simultaneously added a thread for this sequence, it would have collapsed the cardinal κ^+.

4.4.1 $\square_\kappa$ fails

Now we will show that $H_\theta[G] \models \neg\square_\kappa$ by adopting a method by Jensen. First, let us assume towards a contradiction that $\Vdash^{H_\theta[G_\alpha * K]}_{(\dot{P}^1_\kappa)^{G_\alpha * K}} \dot{C}$ is a $\square_\kappa$ sequence. To improve readability, we will abbreviate this as $\Vdash_{P^1_\kappa} \dot{C}$ is a $\square_\kappa$ sequence, and do similar abbreviations whenever the context makes it unambiguous to do so. Furthermore, without loss of generality suppose that $\dot{C} = j_1(\dot{C}')$, that is, that $\dot{C}$ is chosen to be in the range of j_1. Finally, let $C = (\dot{C}^{F^1_\kappa})_\beta \in H_\theta[G]$ and let $C' = (\dot{C}')^{F^1_\alpha} \in H[G']$.

Let let $D = j^{-1}[C_\beta] \in H_\theta[G]$. By elementarity it follows that $(\dot{C}')^{F^1_\alpha}$ is a $\square_\alpha$ sequence with thread D. To see this, let $\xi \in \lim(D) \cap \mathrm{cf}(< \alpha)$. Then $j(\xi) \in \lim j(D) \subseteq \lim(C_\beta)$, so by coherency $C_j(\xi) = C_\beta \cap j(\xi)$. Taking the preimage we get $C'_\xi = D \cap \xi$.

Observe that our iteration above $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$ is $\leq \alpha^+$-distributive. Therefore this segment of our iteration does not add small sets, and so $D \in H_\theta[G_\alpha * F^1_\alpha * F^2_\alpha]$. Next, observe that the iteration P_α has the $< \alpha$ chain condition and therefore does not collapse α^+. As before, this means that forcing with P_α cannot add a thread to a $\square_\alpha$ sequence, so $D \notin H_\theta[G_\alpha]$. Finally we force that $H_\theta[G_\alpha * F^1_\alpha] \models \square_{\alpha,2}$ and thus must preserve α^+. We conclude that $D \notin H_\theta[G_\alpha * F^1_\alpha]$, and so it must be added by the forcing $(\dot{P}^2_\alpha)^{F^!_\alpha}$.

Briefly let us define some notation to use. For any condition $p \in (\dot{P}^2_\alpha)^{F^1_\alpha}$ let $p^* = \langle \bigcup F^1_\alpha \restriction (\sup(p) + 1, \dot{p}\rangle$ be the unique condition in Q_α with upper part $\dot{p}$ and lower part which is the initial segment of the added $\square_{\alpha,2}$ sequence which has just added p.

Now since D is added by the forcing $(\dot{P}^2_\alpha)^{F^!_\alpha}$, there must exist some ordinal $\nu < \alpha^+$ and forcing conditions $s, s' \in P^2_\alpha$ for which

$$s^* \Vdash_{Q_\alpha} \check{\nu} \in \dot{D}, \qquad s'^* \Vdash_{Q_\alpha} \check{\nu} \notin \dot{D},$$

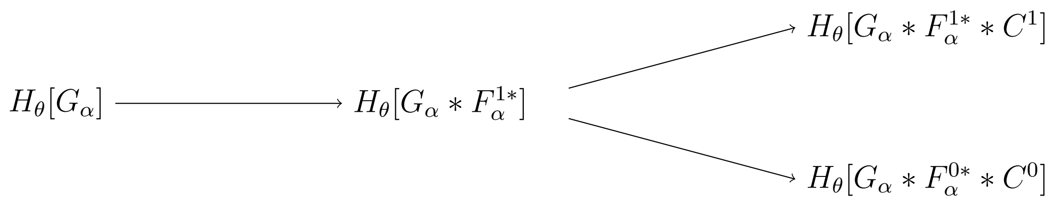

Figure 4.1: The forcing extensions.

where $\dot{D}$ is a Q_α-name for D. That is, let the conditions s and s' witness that the statement "$\check{\nu} \in \dot{D}$" is undecided in the ground model $H_\theta[G_\alpha * F^1_\alpha]$.

We will now build four ω-sequences via induction. First let $s^0_0 = s$, $s^1_0 = s'$, and $\xi^0_0 = \xi^1_0 = \nu$. Now suppose that the i^{th} terms of each sequence have been defined for some $i \in \omega$. Let $\xi^0_{i+1} \geq \xi^1_i$, and let $s^0_{i+1} \Vdash \check{\xi}^0_{i+1} \in \dot{D}$, where $s^0_{i+1} \leq s^0_i$. Next let $\xi^1_{i+1} \geq \xi^0_{i+1}$, and let $s^1_{i+1} \Vdash \check{\xi}^1_{i+1} \in \dot{D}$, where $s^1_{i+1} \leq s^1_i$.

Our induction is complete, so let $s^h = \bigcup s^h_i$ and let $\xi = \bigcup \xi^h_i$ for each $h \in 2$. If necessary, extend s^0 and s^1 such that their domains are both equal to an ordinal $\eta \geq \xi$. Observe that since our two ordinal sequences are interleaved their suprema are equal to ξ for either choice of h. Let $p = F^1_\alpha \restriction \eta \cup \{\langle \eta, \{s^0, s^1\}\rangle\}$ be a condition in P^1_α. For each h we have that $\langle p, s^h\rangle \leq_{Q_\alpha} s^{h*}_i$ for every $i \in \omega$, and thus the condition forces the corresponding decision about the membership of ν in D.

We will now select generics. Let F^{1*}_α be a generic for P^1_α over $H_\theta[G_\alpha]$ containing p. That is, F^{1*}_α agrees with F^1_α at least up through η. Let us call $C^* = (\dot{C}')^{F^{1*}_\alpha} \in H_\theta[G_\alpha * F^{1*}_\alpha]$. Let C^h be $(\dot{P}^2_\alpha)^{F^{1*}_\alpha}$ generics over $H_\theta[G_\alpha * F^{1*}_\alpha]$ containing s^h for each $h \in 2$, respectively. Let us call $D^h = (\dot{D})^{F^1_\alpha * C^h}$ for each $h \in 2$. Because $\xi \in \lim(D^h)$ for each h we have that $\nu \in D^0 \cap \xi = C^*_\xi = D^1 \cap \xi \not\ni \nu$.

This is a contradiction, since the set C^*_ξ is in the ground model $H_\theta[G_\alpha * F^{1*}_\alpha]$, and so the forcing statement "$\check{\nu} \in \dot{D}$" must have already been decided in this model. Therefore our initial assumption that the existence of a square sequence could be forced was false, so $H_\theta[G] \models \neg\square_\kappa$.

4.4.2 κ is measurable

We now need to show that our preparation forcing has preserved the measurability of κ, so that we can next do a Prikry forcing. First, observe that the forcing P^1_κ over $H_\theta[G_\kappa]$ is $(\kappa+1)$-strategically closed and therefore does not add new subsets of κ. Thus in order to show $H_\theta[G] \models \kappa$ is measurable it suffices to show that $H_\theta[G_\kappa] \models \kappa$ is measurable.

Recall that we have the ultrapower map $e : V \to N \cong \mathrm{Ult}(V, U)$ and that $\kappa \notin e(S_\kappa)$, and so the forcing $e(P_\kappa)$ is trivial at the κ step. We may factor $e(P_\kappa)$ as $P_\kappa * \dot{P}_{>\kappa}$ where $\dot{P}_{>\kappa}$ is a name for the Easton support forcing by the same definition but with support in $e(S_\kappa) \cap [\kappa, e(\kappa))$. Finally, let $Q = \dot{P}^{G_\kappa}_{>\kappa}$ be the evaluation of this name by G_κ in the ultrapower N. We are going to construct a generic for Q over $N[G_\kappa]$ which will be defined in $V[G_\kappa]$.

Let λ be the least ordinal in $e(S_\kappa)\backslash\kappa$. It follows that $\lambda \in (\kappa^+, e(\kappa))$ since by elementarity, $e(S_\kappa)$ is cofinal in $e(\kappa)$, and also $(\kappa^+)^N = \kappa^+$, yet λ must be inaccessible in N. Since N is an ultrapower $2^\kappa < e(\kappa) < (2^\kappa)^+$, so this means that since we have assumed GCH, in V we have that λ has cardinality κ^+. Since λ is the least ordinal in which Q has nontrivial support, in $N[G_\kappa]$ we have that Q is $<\lambda$-closed. Additionally, observe that Q has $(e(\kappa)^{++})^N$-many dense subsets.

Since $|e(\kappa)^{++}| = \kappa^+$, $V[G_\kappa]$ sees that Q has κ^+-many dense subsets, and that since $\lambda \in (\kappa^+, \kappa^{++})$, $V[G_\kappa]$ sees that Q is $<\kappa^+$-closed. We will be doing induction to create a κ^+-sequence of conditions from all of our dense sets. For this argument we will need to be able to extend κ-length sequences, and to do so we introduce the following claim which says that $N[G_\kappa]$ contains all κ-length sequences of its elements which are constructed in $V[G_\kappa]$.

<u>Claim</u> 4 ${}^\kappa N[G_\kappa] \cap V[G_\kappa] \subseteq N[G_\kappa]$.

<u>Proof of claim:</u> It is enough to show that $N[G_\kappa]$ is closed under functions into the ordinals, so let $f : \kappa \to \mathrm{On}$ be a function in $V[G_\kappa]$, let $\dot{f} \in P_\kappa^{N[G_\kappa]}$ be a name for f, and let $p \in G_\kappa$ be such that $p \Vdash \dot{f}$ is a function from κ into the ordinals. For each $\alpha < \kappa$ let

$$A_\alpha = \{q \leq p \mid \exists\beta\, q \Vdash \dot{f}(\alpha) = \beta\}.$$

Each A_α is dense below p, so for each $\alpha < \kappa$ and each $q \in A_\alpha$ let

$$g(\alpha, q) = \text{the unique } \beta \text{ for which } q \Vdash \dot{f}(\alpha) = \beta.$$

Now since $|g| = \kappa \cdot |P_\kappa| = \kappa^2 = \kappa$ and ${}^\kappa N \subseteq N$, we have that $g \in N$. Therefore we may define f in $N[G_\kappa]$ by

$$f(\alpha) = \text{the unique } \beta \text{ for which there is some } q \in G_\kappa \text{ s.t. } g(\alpha, q) = \beta.$$

$\square$ (Claim)

We are now ready to do the induction. Let $\langle D_\xi \mid \xi < \kappa^+ \rangle$ be an enumeration of all dense sets $D_\xi \in \mathcal{P}(Q) \cap N[G_\kappa]$. Choose $p_0 \in D_0$. Given some $\alpha < \kappa^+$ and p_α, use the density of $D_{\alpha+1}$ to choose $p_{\alpha+1} \in D_{\alpha+1} \downarrow p_\alpha$. Now suppose that we have defined $\langle p_\xi \mid \xi < \gamma \rangle$ for some $\gamma < \kappa^+$. By our claim this sequence is a set of $N[G_\kappa]$, $(Q \text{ is } < \lambda\text{-closed})^{N[G_\kappa]}$, and in $N[G_\kappa]$ we have that $\lambda > \kappa^+$ is a cardinal. Therefore in $N[G_\kappa]$ we know there exists some p'_γ which lies below p_ξ for all $\xi < \gamma$. Again using density, let $p_\gamma \in D_\gamma \downarrow p'_\xi$. This construction produces the full sequence $\langle p_\xi \mid \xi < \kappa^+ \rangle$ in $V[G_\kappa]$.

Let $G_{>\kappa}$ be the filter generated by $\langle p_\xi \mid \xi < \kappa^+ \rangle$. It meets every dense set in $N[G_\kappa]$ of Q, so it is generic for Q over $N[G_\kappa]$. Finally, observe that our factoring trivially lets us extend e to an elementary embedding $e' : V[G_\kappa] \to N[G_\kappa * G_{>\kappa}]$, so we can derive a measure U' from e'. But this means that U' can be defined from the parameters U, P_κ, G_κ, and $G_{>\kappa}$. Since our above work showed that we can construct $G_{>\kappa}$ in $V[G_\kappa]$ and we know that the rest of our parameters are in the same model, we have shown that $V[G_\kappa]$ contains a measure on κ.

Finally, recall that $H_\theta[G_\kappa] = (H_\theta)^{V[G_\kappa]}$. Since $U' \in V[G_\kappa]$ and $|\mathrm{trcl}(U')| = 2^\kappa < \theta$, we have that $U' \in H_\theta[G_\kappa]$. By our first comment in this section this means that κ remains measurable in $H_\theta[G]$.

4.5 Prikry forcing

We have demonstrated that $H_\theta[G] \models \kappa$ is measurable, so we are now in a position to force κ to be a singular cardinal. Observe that $H[G'] \models \alpha$ is a measurable cardinal as well. Let R be a Prikry forcing on κ and R' be a Prikry forcing on α such that $j(R') = R$. We would like to prove that for any generic K for R we have that $H_\theta[G * K] \models \square_{\kappa,2} + \neg\square_\kappa$. That $\square_{\kappa,2}$ must hold is the same argument we have seen before. Prikry forcing preserves all cardinals, and if forcing by R adds a thread it must collapse κ^+.

What remains to show is that $\square_\kappa$ does not hold in our extension. We will do so with an argument very similar to that of section 4.4.1. However, observe that this time we can not lift j to $H_\theta[G * K]$, since in the extension κ is singular yet the critical point of j must be regular. Our argument therefore will require more care, as we must work in the forcing language when achieving our contradiction.

Let us note now that forcing with R' over $H[G_\alpha * F^1_\alpha]$ is equivalent to forcing over $V[G_\alpha * F^1_\alpha]$ for most statements that we will be concerned with in this section. We begin by assuming towards a contradiction that there is a condition $(s, A) \in R'$ such that

$$j(s, A) \Vdash_R^{H_\theta[G]} \dot{C} \text{ is a } \square_\kappa \text{ sequence,}$$

where $\dot{C} = j(\dot{C}')$ for some R'-name $\dot{C}'$. By elementarity we also have that $(s, A) \Vdash_{R'}^{H[G']}$ $\dot{C}'$ is a $\square_\alpha$ sequence. Since it is forced that $\dot{C}'$ is a $\square_\alpha$ sequence, it will be forced that the order type of each $\dot{C}'_\xi$ is less than α. Thus to each $\xi \in \lim(\alpha, \alpha^+)$, let $(s_\xi, A_\xi) \leq (s, A)$ and γ_ξ be such that $(s_\xi, A_\xi) \Vdash_{R'} \mathrm{ot}(\dot{C}'_\xi) = \check{\gamma}_\xi$. Observe that there are α^+-many such conditions, but there are only α-many possible stems for conditions in R'. Let $\bar{s}$ be a stem such that $S = \{\xi < \alpha^+ \mid s_\xi = \bar{s}\}$ is stationary in $\alpha+$ in $V[G_\alpha * F^1_\alpha]$. Lastly, let $\bar{A} = \bigcap_{\xi \in S} j(A_\xi)$, and observe by κ-completeness this is a set in the ultrafilter used to build R. We now have a single condition $(\bar{s}, \bar{A}) \in R$ such that for every $\xi \in S$, $(\bar{s}, \bar{A}) \leq j(s_\xi, A_\xi)$. Since the ordinals γ_ξ are below the critical point of j, it follows by elementarity that for all $\xi \in S$, $(\bar{s}, \bar{A}) \Vdash_R \mathrm{ot}(\dot{C}_{j}(\xi)) = \check{\gamma}_\xi < \check{\alpha}$.

We now work through a series of claims.

<u>Claim</u> 5 *There is a condition* $(\bar{s}, \bar{A}^*) \leq^*_R (\bar{s}, \bar{A})$ *and a unique set* $D \in H_\theta[G]$ *club in* β *for which*

$$(\bar{s}, \bar{A}^*) \Vdash_R \check{D} \subseteq \dot{C}_\beta.$$

<u>Proof of claim:</u> Let $E = \langle e_\xi \mid \xi < \alpha^+ \rangle \in V$ be an enumeration of a set club in β of order type α^+. Define the formula $\varphi(\xi) :=$ "$\check{e}_\xi \in \dot{C}_\beta$", and let p_0 be a direct extension of $(\bar{s}, \bar{A})$ deciding $\varphi(0)$. If we have p_ξ defined let $p_{\xi+1} \leq^* p_\xi$ decide $\varphi(\xi + 1)$. Suppose that $\gamma < \alpha^+$ is a limit ordinal and that we have already defined the sequence $\langle p_\xi \mid \xi < \gamma \rangle$. Since $\leq^*_R$ is κ-closed, we may pick some p'_γ such that $p'_\gamma \leq^* p_\xi$ for all $\xi < \gamma$. We then let $p_\gamma \leq^* p'_\gamma$ be a

condition that decides $\varphi(\gamma)$. Finally, let $p_{\alpha^+} \leq^* p_\xi$ for all $\xi < \alpha^+$. Since we have only taken direct extensions, we know that p_{α^+} is of the form $(\bar{s}, \bar{A}^*)$ for some $\bar{A}^* \subseteq \bar{A}$. In our ground model let

$$D = \{e_\xi \mid (\bar{s}, \bar{A}^*) \Vdash_R \varphi(\xi)\}.$$

Since $(\bar{s}, \bar{A}^*) \Vdash_R \check{D} = \dot{C}_\beta \cap \check{E}$ and being club is an absolute property it must be that D is club in β.

□ (Claim)

Claim 6 *Define the set $D^{'} = j^{-1}[D] \in H_\theta[G]$. Then $D^{'} \notin V[G_\alpha * F^1_\alpha]$.*

Proof of claim: Suppose otherwise, that $D^{'} \in V[G_\alpha * F^1_\alpha]$. Take any $\xi \in \lim(D^{'}) \cap S$ for which $(\mathrm{cf}(\xi) < \alpha)^{H[G_\alpha * F^1_\alpha]}$. By writing $\xi = \bigcup_{\eta < \mathrm{cf}(\xi)} \xi_\eta$ observe that $j(\xi) = j(\bigcup_{\eta<\mathrm{cf}(\xi)} \xi_\eta) = \bigcup_{\eta<\mathrm{cf}(\xi)} j(\xi_\eta) = \sup j[\xi]$. Since $D^{'} \cap \xi$ is cofinal in ξ, $j[D^{'} \cap \xi]$ is cofinal in $\sup j[\xi] = j(\xi)$. Therefore $\sup j[D^{'} \cap \xi] = j(\xi)$, and so $j(\xi) \in \lim(D)$. By claim 5, since $(\bar{s}, \bar{A}^*)$ forces that D is a subset of C_β, we have that $(\bar{s}, \bar{A}^*) \Vdash_R j(\xi) \in \lim(\dot{C}_\beta)$.

We conclude that $(\bar{s}, \bar{A}^*) \Vdash_R \dot{C}_\beta \cap j(\xi) = \dot{C}_{j(\xi)}$. By intersecting the left side of this forced statement with $\check{D}$, we get

$$(\bar{s}, \bar{A}^*) \Vdash_R \check{D} \cap j(\xi) \subseteq \dot{C}_{j(\xi)},$$

and therefore that

$$(\bar{s}, \bar{A}^*) \Vdash_R \mathrm{ot}(\check{D} \cap j(\xi)) \leq \mathrm{ot}(\dot{C}_{j(\xi)}) = \check{\gamma}_\xi < \check{\alpha}.$$

However since order type is absolute, $V[G_\alpha * F^1_\alpha] \models \mathrm{ot}(D \cap j(\xi)) < \alpha$, and so

$$\mathrm{ot}(D^{'} \cap \xi) = \mathrm{ot}(j[D^{'} \cap \xi]) \leq \mathrm{ot}(D \cap j(\xi)) < \alpha.$$

But this holds for cofinally-many $\xi < \alpha^+$, so we must conclude that $\mathrm{ot}(D^{'}) \leq \alpha$. However, $V[G_\alpha * F^1_\alpha]$ sees that α^+ has not yet been collapsed to α, whereas $D^{'}$ is a set cofinal in α^+ of order type at most α, a contradiction.

□ (Claim)

Claim 7 *If $\xi \in \lim(D^{'})$ is such that $(cf(\xi) < \alpha)^{H[G_\alpha * F^1_\alpha]}$ then there is a single condition $(\bar{s}, B) \in R^{'}$ such that*

$$\eta \in D^{'} \cap \xi \to (\bar{s}, B) \Vdash^{V[G_\alpha * F^1_\alpha]}_{R^{'}} \check{\eta} \in \dot{C}^{'}_\xi$$

and

$$\eta \notin D' \cap \xi \rightarrow (\bar{s}, B) \Vdash^{V[G_\alpha * F^1_\alpha]}_{R'} \check{\eta} \ \notin \dot{C}'_\xi.$$

Proof of claim: Because we know that $j(\xi) \in \lim(D)$, $j(\xi)$ will be forced to be a limit point of C_β, and so $(\bar{s}, \bar{A}^*) \Vdash_R \dot{C}_\beta \cap j(\xi) = \dot{C}_{j(\xi)}$. If we assume that $\eta \in D' \cap \xi$ we know that $j(\eta) \in D \cap j(\xi)$, so $(\bar{s}, \bar{A}^*) \Vdash_R j(\eta) \in \dot{C}_{j(\xi)}$. By elementarity there exists a condition $(\bar{s}, B') \in R'$ such that $(\bar{s}, B') \Vdash_{R'} \eta \in \dot{C}'_\xi$. We may repeat the proof beginning with the assumption that $\eta \notin D \cap \xi$, and in doing so we will get a condition $(\bar{s}, B'') \in R'$ such that $(\bar{s}, B'') \Vdash_{R'} \eta \notin \dot{C}'_\xi$. Let $B = B' \cap B''$.

$\square$ (Claim)

4.5.1 $\square_\kappa$ fails after Prikry forcing

We can now do an argument which is similar to that of section 4.4.1. Let $(p, \dot{q}) \in P^1_\alpha * \dot{P}^2_\alpha$ be a condition that forces claims 6 and 7. Observe that by the closure of our iteration above the α stage $D' \in V[G_\alpha * F^1_\alpha * F^2_\alpha]$. Therefore by claim 6 the set D' is added by the forcing P^2_α. Let $\dot{D}$ be a Q_α-name for D and let $\dot{D}^*$ be a P^2_α-name for D. There must exist an ordinal $\nu < \alpha^+$ and conditions $s, s' \in P^2_\alpha$ such that $s, s' \leq q$ and

$$s^* \Vdash_{Q_\alpha} \check{\nu} \in \dot{D}, \qquad s'^* \Vdash_{Q_\alpha} \check{\nu} \notin \dot{D}.$$

As before we will build four ω-sequences via induction. First let $s^0_0 = s$, $s^1_0 = s'$, and $\xi^0_0 = \xi^1_0 = \nu$. Now suppose that the i^{th} terms of each sequence have been defined for some $i \in \omega$. Let $\xi^0_{i+1} \geq \xi^1_i$, and let $s^0_{i+1} \Vdash \check{\xi}^0_{i+1} \in \dot{D}$, where $s^0_{i+1} \leq s^0_i$. Next let $\xi^1_{i+1} \geq \xi^0_{i+1}$, and let $s^1_{i+1} \Vdash \check{\xi}^1_{i+1} \in \dot{D}$, where $s^1_{i+1} \leq s^1_i$.

Our induction is complete, so let $s^h = \bigcup s^h_i$ and let $\xi = \bigcup \xi^h_i$ for each $h \in 2$. If necessary, extend s^0 and s^1 such that their domains are both equal to an ordinal $\eta \geq \xi$. Observe that since our two ordinal sequences are interleaved their suprema are equal to ξ for either choice of h. Let $p = F^1_\alpha \restriction \eta \cup \{\langle \eta, \{s^0, s^1\}\rangle\}$ be a condition in P^1_α. For each h we have that $\langle p, s^h \rangle \leq_{Q_\alpha} s^{h*}_i$ for every $i \in \omega$, and thus the condition forces the corresponding decision about the membership of ν in D.

We will now select generics. Let F^{1*}_α be a generic for P^1_α over $H_\theta[G_\alpha]$ containing p. That is, F^{1*}_α agrees with F^1_α at least up through η. Let us call $C^* = (\dot{C}')^{F^{1*}_\alpha} \in H_\theta[G_\alpha * F^{1*}_\alpha]$. Let

C^h be $(\dot{P}^2_\alpha)^{F^{1*}_\alpha}$ generics over $H_\theta[G_\alpha * F^{1*}_\alpha]$ containing s^h for each $h \in 2$, respectively. Let us call $D^h = (\dot{D})^{F^1_\alpha * C^h}$ for each $h \in 2$.

Here our argument differs from the work in section 4.4.1. Observe that we have $\nu \in D^0$ and $\nu \notin D^1$, and also that $\xi \in \lim(D^h)$ for each $h \in 2$. Since the conditions $(p, \dot{q}^h)$ for each h force claim 7, we have that in $V[G_\alpha * F^{1*}_\alpha * C^0]$

$$(\bar{s}, \bar{A}^*) \Vdash^{V[G_\alpha * F^{1*}_\alpha]}_{R'} \check{\nu} \in \dot{C}_\xi,$$

and in $V[G_\alpha * F^{1*}_\alpha * C^1]$ we have that

$$(\bar{s}, \bar{A}^*) \Vdash^{V[G_\alpha * F^{1*}_\alpha]}_{R'} \check{\nu} \notin \dot{C}_\xi.$$

Observe that although claim 7 was stated using a generic F^1_α all we needed was any generic for P^1_α, so the above statements hold using F^{1*}_α. But since the above forcing statements are forced by the same condition over the same model, we have a contradiction.

4.6 Conclusion

Thus beginning with a cardinal which is both subcompact and measurable, we have forced a model in which $\square_{\kappa,2}$ holds while simultaneously $\square_\kappa$ fails for a singular cardinal κ.

There are some natural extensions of this result which we may be able to produce with identical or slightly stronger assumptions. Namely, it should be easy to get the proof to go through for $\square_{\kappa,n+1} + \neg\square_{\kappa,n}$ at singular κ and arbitrary $n \in \omega$ without changing our hypothesis. On the other hand, we expect to be able to adopt our methods in this paper's first result to get the result at $\aleph_\omega$, but this may require a slight strengthening of our subcompactness assumption.

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