25 Practice Problems for Derivatives

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1 Power rule

Find the derivative of each of the following functions:

1. $a(x) = 3x^7 - 10x + 24x^3 - 1$

$$a^{'}(x) = 3 \cdot 7x^6 - 10 \cdot 1x^0 + 24 \cdot 3x^2 - 0 = 21x^6 - 10 + 72x^2$$

2. $b(x) = \frac{x^2-5x+3}{\sqrt[3]{x}}$

$$b(x) = \frac{x^2 - 5x + 3}{x^{1/3}} = (x^2 - 5x + 3)\, x^{-1/3} = x^{5/3} - 5x^{2/3} + 3x^{-1/3}$$

$$b^{'}(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} - x^{-4/3}$$

3. $c(x) = 5^2 - \frac{3}{x^4}$

$$c(x) = 25 - 3x^{-4}$$

$$c^{'}(x) = 0 - 3 \cdot (-4x^{-5}) = 12x^{-5}$$

2 Product rule

Find the derivative of each of the following functions:

1. $d(x) = \sin(x)\cos(x)$

$$d^{'}(x) = \cos(x)\cos(x) + \sin(x) \cdot (-\sin(x)) = \cos^2(x) - \sin^2(x)$$

2. $e(x) = x^{10}e^x$

$$e^{'}(x) = 10x^9e^x + x^{10}e^x$$

3. $f(x) = 2^x x^2 \tan(x)$

$$f^{'}(x) = (2^x \ln(2))\, x^2 \tan(x) + 2^x\, (2x)\, \tan(x) + 2^x x^2 \left(\sec^2(x)\right)$$

3 Quotient rule

Find the derivative of the following function:

- $g(x) = \frac{2e^x+4}{\cos(x)+3x-1}$

$$g^{'}(x) = \frac{(2e^x)(\cos(x) + 3x - 1) - (2e^x + 4)(-\sin(x) + 3)}{(\cos(x) + 3x - 1)^2}$$

Find the <u>second</u> derivative of the following function:

- $h(x) = \frac{\sin(x)}{x^2}$

$$h^{'}(x) = \frac{\cos(x)\, x^2 - \sin(\text{x})\, 2\text{x}}{x^4}$$

$$h^{''}(x) = \frac{\left(\left(-\sin(x)\, x^2 + \cos(x)\, 2x\right) - (\cos(x)\, 2x + \sin(x)\, 2)\right) x^4 - \left(\cos(x)\, x^2 - \sin(\text{x})\, 2\text{x}\right) 4x^3}{x^8}$$

4 Chain rule

For each of the following, write the given function as a composition of two functions, i.e., as $f(g(x))$, where you have identified f and g. Then take the derivative using the chain rule. Note: some problems may require more than one chain rule.

1. $i(x) = (22x^4 + \sqrt{x})^9$

> Let $f(u) = u^9$, and $u = g(x) = 22x^4 + \sqrt{x} = 22x^4 + x^{1/2}$. Then $i(x) = f(g(x))$. Also, $f^{'}(u) = 9u^8$, and $g^{'}(x) = 88x^3 + \frac{1}{2}x^{-1/2}$. So by the chain rule,
>
> $$i^{'}(x) = f^{'}(g(x)) \cdot g^{'}(x) = 9\left(22x^4 + x^{1/2}\right)^8 \cdot \left(88x^3 + \frac{1}{2}x^{-1/2}\right)$$

2. $j(x) = \sin(x^3 + 1)$

> Let $f(u) = \sin(u)$, and $u = g(x) = x^3 + 1$. Then $j(x) = f(g(x))$. Also, $f^{'}(u) = \cos(u)$, and $g^{'}(x) = 3x^2$. So by the chain rule,
>
> $$j^{'}(x) = f^{'}(g(x)) \cdot g^{'}(x) = \cos\left(x^3 + 1\right) 3x^2$$

3. $k(x) = \sin^3(x) + 1$

> Let , $f(u) = u^3 + 1$ and $u = g(x) = \sin(x)$. Then $k(x) = f(g(x))$. Also, $f^{'}(u) = 3u^2$ and $g^{'}(x) = \cos(x)$. So by the chain rule,
>
> $$k^{'}(x) = f^{'}(g(x)) \cdot g^{'}(x) = 3\sin^2(x)\cos(x)$$
>
> <u>Easier solution</u>: It is probably better to think of $\sin^3(x)$ and 1 as different functions that we are adding together, and then just do chain rule on $\sin^3(x)$. Then $f(u) = u^3$ and $u = g(x) = \sin(x)$. So we get:
>
> $$k^{'}(x) = \frac{d}{dx}\sin^3(x) + \frac{d}{dx}1 = 3\sin^2(x)\cos(x)$$

4. $l(x) = \ln(\sin(x))$

> Let $f(u) = \ln(u)$, and $u = g(x) = \sin(x)$. Then $l(x) = f(g(x))$. Also, $f^{'}(u) = \frac{1}{u}$, and $g^{'}(x) = \cos(x)$. So by the chain rule,
>
> $$l^{'}(x) = f^{'}(g(x)) \cdot g^{'}(x) = \frac{1}{\sin(x)}\cos(x) = \frac{\cos(x)}{\sin(x)} = \cot(x)$$

5. $m(x) = \sin\left(\cos\left(e^{5x^2 - 3x + 2}\right)\right)$

This time we have many nested chain rules. It will complicate things to give names to each function that we are composing. Instead, let's simply **work from the outside, inwards**. Remember, we **take the derivative of one function at a time**, and then plug in the next function(s), unchanged. Then move one step inwards.

$$m'(x) = \cos\left(\cos\left(e^{5x^2-3x+2}\right)\right) \cdot \left(-\sin\left(e^{5x^2-3x+2}\right)\right) \cdot \left(e^{5x^2-3x+2}\right) \cdot (10x-3)$$

5 Multiple rules

Find the derivative of each of the following functions:

1. $n(x) = \frac{x\ln(x)}{x^{3/2}+1}$

$$n'(x) = \frac{\left(1 \cdot \ln(x) + x \cdot \frac{1}{x}\right)\left(x^{3/2}+1\right) - (x\ln(x))\,\frac{3}{2}x^{1/2}}{\left(x^{3/2}+1\right)^2}$$

2. $o(x) = (3x^5 - 2x + 7)^{13}e^x$

$$o'(x) = 13(3x^5-2x+7)^{12}(15x^4-2)e^x + (3x^5-2x+7)^{13}e^x$$

3. $p(x) = \ln\left(x - \frac{1}{e^x}\right)$

$$p(x) = \ln\left(x - e^{-x}\right)$$

$$p'(x) = \frac{1}{x-e^{-x}}\left(1 - e^{-x}\cdot(-1)\right) = \frac{1+e^{-x}}{x-e^{-x}}$$

4. $q(x) = \ln\left(x\sin(x)\right)$

$$q'(x) = \frac{1}{x\sin(x)}\left(\sin(x) + x\cos(x)\right)$$

6 Implicit differentiation

Find y' in each of the following examples. Remember, y **is a function**! This means you must use some extra derivative rules. Golden rule: <u>if your derivative of a $y-$term doesn't have $y^{'}$, you missed a derivative rule</u>!

1. $x^2y + \sin(y) = 5y^2 + 3$

$$\left(2xy + x^2y^{'}\right) + \cos(y)\,y^{'} = 10yy^{'}$$

$$x^2y^{'} + \cos(y)\,y^{'} - 10yy^{'} = -2xy$$
$$y^{'}\left(x^2 + \cos(y) - 10y\right) = -2xy$$
$$y^{'} = \frac{-2xy}{x^2 + \cos(y) - 10y}$$

2. $e^{2y+1} = x$

$$e^{2y+1} \cdot 2y^{'} = 1$$

$$y^{'} = \frac{1}{2}e^{-(2y+1)}$$

Find $y^{''}$ from the following equation.

- $x^3 + y^3 = 1$

Take an implicit derivative with respect to x:

$$3x^2 + 3y^2y^{'} = 0$$

Do it again:

$$6x + \left(\left(6yy^{'}\right)y^{'} + 3y^2y^{''}\right) = 0$$

$$6x + 6y\left[y^{'}\right]^2 + 3y^2y^{''} = 0$$

The final answer can be in terms of x and y, but not $y^{'}$. Look back to our first differentiated equation, and solve for $y^{'}$:

$$3x^2 + 3y^2y^{'} = 0$$
$$3y^2y^{'} = -3x^2$$
$$y^{'} = -\frac{x^2}{y^2}$$

Now plug into our second differentiated equation:

$$6x + 6y\left[y^{'}\right]^2 + 3y^2y^{''} = 0$$

$$6x + 6y\left(-\frac{x^2}{y^2}\right)^2 + 3y^2y'' = 0$$

$$6x + 6\frac{x^4}{y^3} + 3y^2y'' = 0$$

$$3y^2y'' = -6x - 6\frac{x^4}{y^3}$$

$$y'' = \frac{-6x - 6\frac{x^4}{y^3}}{3y^2}$$

7 Logarithmic differentiation

In the following problems you will find it helpful to make an equation of the form $y = ...$ and take a natural logarithm of both sides before differentiating.

1. $r(x) = x^x$

$$\text{Let } y = x^x$$

$$\ln(y) = \ln x^x$$

$$\ln(y) = x\ln(x)$$

Differentiate.

$$\frac{y'}{y} = 1 \cdot \ln(x) + x \cdot \frac{1}{x}$$

$$y' = y\left(\ln(x) + 1\right)$$

Substitute in for y.

$$r'(x) = y' = x^x\left(\ln(x) + 1\right)$$

2. $s(x) = (x^2 - 4)^{\sin(x)}$

$$y = \left(x^2 - 4\right)^{\sin(x)}$$

$$\ln(y) = \ln\left(\mathrm{x}^2 - 4\right)^{\sin(\mathrm{x})}$$

$$\ln(y) = \sin(x)\ln\left(\mathrm{x}^2 - 4\right)$$

$$\frac{y'}{y} = \cos(x)\ln\left(\mathrm{x}^2 - 4\right) + \sin(\mathrm{x})\frac{2x}{x^2 - 4}$$

$$y' = y\left(\cos(x)\ln\left(\mathrm{x}^2 - 4\right) + \sin(\mathrm{x})\frac{2x}{x^2 - 4}\right)$$

$$s'(x) = y' = \left(x^2 - 4\right)^{\sin(x)}\left(\cos(x)\ln\left(\mathrm{x}^2 - 4\right) + \sin(\mathrm{x})\frac{2x}{x^2 - 4}\right)$$

3.

$$t(x) = \frac{\sqrt{4x^3 - x + 1}}{x^{2/3}\tan(x)}$$

$$\ln y = \ln \frac{\sqrt{4x^3 - x + 1}}{x^{2/3}\tan(x)}$$

$$= \ln(4x^3 - x + 1)^{1/2} - \ln\left(x^{2/3}\tan(x)\right)$$

$$= \frac{1}{2}\ln(4x^3 - x + 1) - \left(\ln x^{2/3} + \ln\tan(x)\right)$$

$$= \frac{1}{2}\ln(4x^3 - x + 1) - \frac{2}{3}\ln x - \ln\tan(x)$$

Now differentiate:

$$\frac{y'}{y} = \frac{1}{2} \cdot \frac{12x^2 - 1}{4x^3 - x + 1} - \frac{2}{3} \cdot \frac{1}{x} - \frac{\sec^2(x)}{\tan(x)}$$

$$t'(x) = y' = \frac{\sqrt{4x^3 - x + 1}}{x^{2/3}\tan(x)} \left(\frac{1}{2} \cdot \frac{12x^2 - 1}{4x^3 - x + 1} - \frac{2}{3x} - \frac{\sec^2(x)}{\tan(x)}\right)$$

8 Related rates

1. A spherical snowball is melting in the sun. Its radius is decreasing at a rate of 1 cm/s. When the radius reaches 5 cm, how quickly is the snowball losing volume?

I want you to draw a picture for every related rates problem. You should do so for these problems, although the solutions do not include them. Our equation is the volume equation for a sphere.

$$V = \frac{4}{3}\pi r^3$$

The quantities that we know or want to find are:

$$\frac{dr}{dt} = -1 \text{ cm/s} \qquad \frac{dV}{dt} = ?$$

Differentiate our equation.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi\,(5)^2\,(-1) = -100\pi \text{ cm}^3/\text{s}$$

2. An airplane flies overhead 2 miles up at a speed of 500 mi/hr. When it has travelled 1 mile from where you are, how quickly is the distance from you to the airplane increasing?

Again, draw the picture. If you do, you'll see that we have a right triangle. The triangle has height 2 mi. Let us label its base x, and its hypotenuse D. Our equation is

$$2^2 + x^2 = D^2$$

The quantities that we know or want to find are:

$$\frac{dx}{dt} = 500 \text{ mi/h} \qquad \frac{dD}{dt} = ?$$

Differentiate:

$$2x\frac{dx}{dt} = 2D\frac{dD}{dt}$$

$$\frac{dD}{dt} = \frac{x}{D}\frac{dx}{dt}$$

We need to know the value of D when $x = 1$ mi. Plug into our original equation.

$$2^2 + 1^2 = D^2$$

$$D = \sqrt{5}$$

$$\frac{dD}{dt} = \frac{1}{\sqrt{5}}(500) = \frac{500}{\sqrt{5}} \text{ mi/h}$$